# Chapter 5. Continuity and Differentiability

## Continuity

### 4 Marks Questions

**1.** Find the value of $k$, so that the function $f$ defined below, is continuous at $x = 0$, where

$$f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}.$$

**All India 2014C**

Given function is

$$f(x) = \begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

Also, given $f(x)$ is continuous at $x = 0$.

$\therefore \quad (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) \quad \ldots\text{(i)}$

Now, $\quad \text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{1-\cos 4x}{8x^2}$$

$$= \lim_{h \to 0} \frac{1-\cos(-4h)}{8h^2}$$

[put $x = 0 - h = -h$, when $x \to 0$, $h \to 0$] **(1)**

$$= \lim_{h \to 0} \frac{1-\cos 4h}{8h^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 2h}{8h^2}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{\sin^2 2h}{4h^2} = \lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2 = 1 \quad \textbf{(1)}$$

At $x = 0$, $f(0) = k$

Now, from Eq. (i), we have

$$\text{LHL} = f(0)$$

$$\Rightarrow \quad 1 \cdot 1 = k \Rightarrow k = 1 \quad \textbf{(1)}$$

Hence, for $k = 1$, the given function $f(x)$ is continuous at $x = 0$. **(1)**

**2.** If $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{when } x<0\\ a, & \text{when } x=0\\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{when } x>0\end{cases}$

and $f$ is continuous at $x=0$, then find the value of $a$. **Delhi 2013C**

Given,

$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & \text{when } x<0\\ a, & \text{when } x=0\\ \dfrac{\sqrt{x}}{(\sqrt{16+\sqrt{x}})-4}, & \text{when } x>0\end{cases}$$

Since, $f(x)$ is continuous at $x=0$.

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=f(0)$$

$$\Rightarrow \lim_{x\to 0^-}\frac{1-\cos 4x}{x^2}$$

$$=\lim_{x\to 0^+}\frac{\sqrt{x}}{(\sqrt{16+\sqrt{x}})-4}=f(0) \quad \textbf{(1)}$$

$$\Rightarrow \lim_{h \to 0} \frac{1-\cos 4(0-h)}{(0-h)^2}$$

$$= \lim_{h \to 0} \frac{\sqrt{0+h}}{(\sqrt{16+\sqrt{0+h}}-4)} = f(0)$$

$$\Rightarrow \lim_{h \to 0} \frac{1-\cos 4h}{h^2}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4} \times \frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4} = a \quad \textbf{(1)}$$

$$\Rightarrow \lim_{h \to 0} \frac{2\sin^2 2h}{h^2} = \lim_{h \to 0} \frac{\sqrt{h}(\sqrt{16+\sqrt{h}}+4)}{(16+\sqrt{h})-16} = a$$

$$[\because 1-\cos 2\theta = 2\sin^2\theta \text{ and } (a+b)(a-b) = a^2-b^2]$$

$$\Rightarrow 2\lim_{h \to 0} \left(\frac{\sin 2h}{2h}\right)^2 \times 4$$

$$= \lim_{h \to 0} \frac{\sqrt{h}(\sqrt{16+\sqrt{h}}+4)}{\sqrt{h}} = a \quad \textbf{(1)}$$

$$\left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$\Rightarrow \quad 2 \times (1)^2 \times 4 = \sqrt{16+\sqrt{0}}+4 = a$$

$$\Rightarrow \quad 8 = 4+4 = a$$

$$\therefore \quad a = 8 \quad \textbf{(1)}$$

**3.** Find the value of $k$, for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

is continuous at $x=0$. **All India 2013**

Given, $f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$

is continuous at $x = 0$.

Now, $f(0) = \dfrac{2 \cdot 0 + 1}{0 - 1} = \dfrac{1}{-1} = -1$ **(1)**

$$\text{LHL} = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}$$

$$= \lim_{h \to 0} \frac{\sqrt{1-kh}-\sqrt{1+kh} \times (\sqrt{1-kh}+\sqrt{1+kh})}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$$ **(1)**

$$= \lim_{h \to 0} \frac{(1-kh)-(1+kh)}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \lim_{h \to 0} \frac{-2kh}{-h(\sqrt{1-kh}+\sqrt{1+kh})}$$ **(1)**

$$= \lim_{h \to 0} \frac{2k}{\sqrt{1-kh}+\sqrt{1+kh}} = \frac{2k}{1+1} = \frac{2k}{2} = k$$

$\because$ $f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = \text{LHL} \Rightarrow -1 = k$

$\Rightarrow$ $k = -1$ **(1)**

**4.** Find the value of $k$, so that the function $f$ defined by

$f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x}, & \text{if } x \ne \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$ is continuous at

$x = \dfrac{\pi}{2}$.

**HOTS; Delhi 2012C; Foreign 2011**

> A function $f(x)$ is said to be continuous at point $x = a$, if LHL at $(x = a)$ = RHL at $(x = a) = f(a)$. So, to find the value of $k$, we equate any one of LHL or RHL to $f(a)$ and simplify it.

Given function is $f(x) = \begin{cases} \dfrac{k\cos x}{\pi - 2x}, & \text{if } x \neq \dfrac{\pi}{2} \\ 3, & \text{if } x = \dfrac{\pi}{2} \end{cases}$

Also, given that function is continuous at $x = \pi/2$.

$\therefore$ At $x = \dfrac{\pi}{2}$, LHL = RHL = $f\left(\dfrac{\pi}{2}\right)$ ...(i) **(1)**

Now, LHL = $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \dfrac{k\cos x}{\pi - 2x}$

$$\text{LHL} = \lim_{h \to 0} \frac{k\cos\left(\frac{\pi}{2} - h\right)}{\pi - 2\left(\frac{\pi}{2} - h\right)} = \lim_{h \to 0} \frac{k\sin h}{\pi - \pi + 2h}$$

$\left[\text{put } x = \dfrac{\pi}{2} - h, \text{ when } x \to \dfrac{\pi}{2}, \text{ then } h \to 0\right]$

$\left[\because \cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta\right]$

$$= \lim_{h \to 0} \frac{k\sin h}{2h}$$

$$= \frac{k}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{k}{2} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$\Rightarrow$ LHL $= k/2$ **(1½)**

Also, from the given function, we get

$$f\left(\frac{\pi}{2}\right) = 3$$

**(1/2)**

Now, from Eq. (i), we have

$$\text{LHL} = f\left(\frac{\pi}{2}\right) \Rightarrow \frac{k}{2} = 3$$

Hence, $k = 6$ **(1)**

**5.** Find the value of $a$ for which the function $f$ is defined as $f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$

is continuous at $x = 0$. **Delhi 2011**

Given function is

$$f(x) = \begin{cases} a \sin \frac{\pi}{2}(x+1), & x \le 0 \\ \frac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \text{LHL} = \text{RHL} = f(0) \quad \ldots\text{(i)} \quad \textbf{(1)}$$

Now, $$\text{LHL} = \lim_{x \to 0^-} a \sin \frac{\pi}{2}(x+1)$$

$$\text{LHL} = \lim_{h \to 0} a \sin \frac{\pi}{2}(-h+1)$$

[put $x = 0 - h = -h$]

$$= \lim_{h \to 0} \frac{a \sin\left[\frac{\pi(-h+1)}{2}\right]}{\frac{\pi(-h+1)}{2}} \times \frac{\pi(-h+1)}{2}$$

$\left[\because \text{multiplying numerator and denominator by } \frac{\pi(-h+1)}{2}\right]$

$$= \lim_{h \to 0} \frac{a\pi(-h+1)}{2} \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$\Rightarrow \quad \text{LHL} = \frac{a\pi}{2} \quad \text{[put } h = 0\text{]} \quad \textbf{(1)}$$

Now, $$\text{RHL} = \lim_{x \to 0^+} \frac{\tan x - \sin x}{x^3}$$

Put $x = 0 + h = h$, we get

$$\text{RHL} = \lim_{h \to 0} \frac{\tan h - \sin h}{h^3} = \lim_{h \to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3}$$

$\sin h - \sin h \cos h \quad \sin h(1 - \cos h)$

$$= \lim_{h \to 0} \frac{\sin h - \sin h \cos h}{h^3 \cos h} = \lim_{h \to 0} \frac{\sin h(1 - \cos h)}{h^3 \cos h}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{1 - \cos h}{h^2} \cdot \lim_{h \to 0} \frac{1}{\cos h}$$

$$= 1 \times \lim_{h \to 0} \frac{1 - \cos h}{h^2} \times 1$$

$$\left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \to 0} \frac{1}{\cos h} = \frac{1}{\cos 0} = \frac{1}{1} = 1\right]$$

$$= \lim_{h \to 0} \frac{1 - \cos h}{h^2}$$

$$= \lim_{h \to 0} \frac{2 \sin^2 \frac{h}{2}}{h^2} \quad \left[\because 1 - \cos x = 2 \sin^2 \frac{x}{2}\right]$$

$$= \lim_{h \to 0} \frac{2 \times \sin^2 \frac{h}{2}}{\left(\frac{h^2}{4} \times 4\right)} = \lim_{h \to 0} \frac{2}{4} \times \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\frac{h^2}{4}}$$

[∵ multiplying numerator and denominator by 4]

$$= \frac{1}{2} \times \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 = \frac{1}{2} \times 1 \times 1$$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right] \quad \textbf{(1)}$$

$$\Rightarrow \text{RHL} = \frac{1}{2}$$

Now, from Eq. (i), we have LHL = RHL

$$\therefore \quad \frac{a\pi}{2} = \frac{1}{2} \Rightarrow a\pi = 1 \Rightarrow a = \frac{1}{\pi} \quad \textbf{(1)}$$

**6.** If the function $f(x)$ given by

$$f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1\end{cases}$$

is continuous at $x=1$, then find the values of $a$ and $b$. **Delhi 2011; All India 2010**

The given function is

$$f(x)=\begin{cases}3ax+b, & \text{if } x>1\\ 11, & \text{if } x=1\\ 5ax-2b, & \text{if } x<1\end{cases}$$

Given that $f(x)$ is continuous at $x=1$.

$\therefore \quad \text{LHL} = \text{RHL} = f(x) \quad$ ...(i)

Now, $\text{LHL} = \lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} (5ax-2b)$ **(1)**

$\text{LHL} = \lim_{h\to 0} [5a(1-h)-2b]$

[put $x=1-h$, when $x\to 1, h\to 0$]

$= \lim_{h\to 0} (5a-5ah-2b)$

$\Rightarrow \quad \text{LHL} = 5a-2b$ [put $h=0$] **(1)**

Now, $\text{RHL} = \lim_{x\to 1^+} (3ax+b)$

$\text{RHL} = \lim_{h\to 0} [3a(1+h)+b]$

[put $x=1+h$, when $x\to 1, h\to 0$]

$= \lim_{h\to 0} (3a+3ah+b)$

$\Rightarrow \quad \text{RHL} = 3a+b$ [put $h=0$]

Also, given that $f(1)=11$ **(1)**

Now, from Eq. (i), we have

$\text{RHL} = f(1)$

$$\Rightarrow \quad 3a + b = 11 \quad \text{...(ii)}$$

and $\quad \text{LHL} = f(1)$

$$\Rightarrow \quad 5a - 2b = 11 \quad \text{...(iii)}$$

On multiplying Eq. (ii) by 5 and Eq. (iii) by 3 and then subtracting, we get

$$\begin{array}{r} 15a + 5b = 55 \\ 15a - 6b = 33 \\ - \quad + \qquad - \\ \hline 11b = 22 \end{array}$$

$$\Rightarrow \quad b = 2$$

On putting the value of $b$ in Eq. (ii), we get

$$3a + 2 = 11 \Rightarrow 3a = 9 \Rightarrow a = 3$$

Hence, $a = 3$ and $b = 2$. **(1)**

**7.** Find the values of $a$ and $b$ such that the following function $f(x)$ is a continuous function.

$$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$

**Delhi 2011**

The given function is continuous that means $f(x)$ is continuous in its domain [2,10], so we take $x = 2$ and $x = 10$ to check continuity and then find $a$ and $b$.

The given function is

$$f(x) = \begin{cases} 5, & x \leq 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \geq 10 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 2$ and at $x = 10$.

$\therefore$ By definition,

$$\text{LHL} = \text{RHL} = f(2) \quad \text{...(i)}$$

and $$\text{LHL} = \text{RHL} = f(10) \quad \text{...(ii)} \quad \textbf{(1)}$$

Now, first we calculate LHL and RHL at $x = 2$.

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 5$$

$\Rightarrow$ $$\text{LHL} = 5$$

and $$\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$$

$$\text{RHL} = \lim_{h \to 0} a(2 + h) + b = \lim_{h \to 0} (2a + ah + b)$$

[put $x = 2 + h$, when $x \to 2, h \to 0$]

$$= 2a + b \quad \text{[put } h = 0]$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$\Rightarrow$ $$2a + b = 5 \quad \text{...(iii)} \quad \textbf{(1)}$$

Now, we find LHL and RHL at $x = 10$.

$$\text{LHL} = \lim_{x \to 10^-} f(x) = \lim_{x \to 10^-} (ax + b)$$

$$\text{LHL} = \lim_{h\to 0}[a(10-h)+b]$$

[put $x = 10 - h$, when $x \to 10, h \to 0$]

$$= \lim_{h\to 0} (10a - ah + b)$$

$\Rightarrow$ $\text{LHL} = 10a + b$ [put $h = 0$]

and $\text{RHL} = \lim_{x\to 10^+} f(x) = \lim_{x\to 10^+} 21 = 21$

Now, from Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$\Rightarrow$ $10a + b = 21$ ...(iv) **(1)**

On subtracting Eq. (iv) from Eq. (iii), we get

$$-8a = -16 \Rightarrow a = 2$$

On putting $a = 2$ in Eq. (iv), we get

$$20 + b = 21$$

$\therefore$ $b = 1$

Hence, $a = 2$ and $b = 1$. **(1)**

**8.** Find the relationship between $a$ and $b$, so that the function $f$ defined by

$$f(x) = \begin{cases} ax + 1, & \text{if } x \le 3 \\ bx + 3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$. **All India 2011**

The given function is $f(x) = \begin{cases} ax + 1, \text{ if } x \le 3 \\ bx + 3, \text{ if } x > 3 \end{cases}$

Also, given that $f(x)$ is continuous at point $x = 3$.

$\therefore \quad \text{LHL} = \text{RHL} = f(3) \quad \ldots(i)$ **(1)**

Now, $\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$

$\text{LHL} = \lim_{h \to 0} [a(3 - h) + 1]$

[put $x = 3 - h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (3a - ah + 1)$

$\Rightarrow \quad \text{LHL} = 3a + 1 \quad$ [put $h = 0$] **(1)**

$\text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$

$\text{RHL} = \lim_{h \to 0} [b(3 + h) + 3]$

[put $x = 3 + h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (3b + bh + 3)$

$\Rightarrow \quad \text{RHL} = 3b + 3 \quad$ [put $h = 0$] **(1)**

From Eq. (i), we have

$\text{LHL} = \text{RHL} \Rightarrow 3a + 1 = 3b + 3$

$\Rightarrow 3a - 3b = 2$, which is the required relation between $a$ and $b$. **(1)**

**9.** Find the value of $k$, so that the function $f$ defined by $f(x) = \begin{cases} kx + 1, \text{ if } x \le \pi \\ \cos x, \text{ if } x > \pi \end{cases}$

is continuous at $x = \pi$. **Foreign 2011**

The given function is $f(x) = \begin{cases} kx + 1, \text{ if } x \le \pi \\ \cos x, \text{ if } x > \pi \end{cases}$

Also, given that $f(x)$ is continuous at $x = \pi$.

$\therefore \quad \text{LHL} = \text{RHL} = f(\pi) \quad \ldots(i)$ **(1)**

Now, LHL $= \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^-} (kx + 1)$

LHL $= \lim_{h \to 0} [k(\pi - h) + 1]$

[put $x = \pi - h$, when $x \to \pi, h \to 0$]

$= \lim_{h \to 0} (k\pi - kh + 1)$

$= k\pi + 1$ [put $h = 0$] **(1)**

and RHL $= \lim_{x \to \pi^+} f(x) = \lim_{x \to \pi^+} \cos x$

RHL $= \lim_{h \to 0} \cos(\pi + h)$

[put $x = \pi + h$, when $x \to \pi, h \to 0$]

$= \cos \pi$ [put $h = 0$] **(1)**

$= -1$ [$\because \cos \pi = -1$]

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$$\Rightarrow \quad k\pi + 1 = -1$$

$$\Rightarrow \quad k\pi = -2$$

$$\therefore \quad k = \frac{-2}{\pi}$$ **(1)**

**10.** For what values of $\lambda$, is the function

$$f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$$

is continuous at $x = 0$? **Foreign 2011**

The given function is

$$f(x) = \begin{cases} \lambda (x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1 & , \text{if } x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore \quad LHL = RHL = f(0) \quad$ ...(i)

Now, $LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lambda (x^2 - 2x)$ **(1)**

$$LHL = \lim_{h \to 0} \lambda (h^2 + 2h)$$

[put $x = 0 - h = -h$, when $x \to 0, h \to 0$]

$= \lambda (0) \quad$ [put $h = 0$] **(1)**

$= 0$

and $\quad RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 1)$

$$RHL = \lim_{h \to 0} (4h + 1)$$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$\Rightarrow \quad RHL = 1 \quad$ [put $h = 0$] **(1)**

Thus, $\quad LHL \ne RHL$

But it is given that $LHL = RHL \quad$ [from Eq. (i)]

Therefore, we get a contradiction.

Hence, there doesn't exist any real value of $\lambda$ for which $f(x)$ is continuous at $x = 0$. **(1)**

**11.** Discuss the continuity of the function $f(x)$ at $x = 1/2$, when $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 1/2 + x, & 0 \le x < 1/2 \\ 1, & x = 1/2 \\ 3/2 + x, & 1/2 < x \le 1 \end{cases}$$

Delhi 2011C

Here, we find LHL, RHL and $f\left(\frac{1}{2}\right)$. If $LHL = RHL = f\left(\frac{1}{2}\right)$, then we say that $f(x)$ is continuous at $x = \frac{1}{2}$, otherwise $f(x)$ is discontinuous at $x = \frac{1}{2}$.

The given function is

$$f(x) = \begin{cases} \frac{1}{2} + x, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ \frac{3}{2} + x, & \frac{1}{2} < x \le 1 \end{cases}$$

We have to check continuity of $f(x)$ at $x = \frac{1}{2}$.

**(1)**

Now, $\text{LHL} = \lim_{x \to \frac{1}{2}^-} f(x) = \lim_{x \to \frac{1}{2}^-} \left(\frac{1}{2} + x\right)$

$$\text{LHL} = \lim_{h \to 0} \left(\frac{1}{2} + \frac{1}{2} - h\right)$$

$\left[\text{put } x = \frac{1}{2} - h, \text{ when } x \to \frac{1}{2}, h \to 0\right]$

$$= \lim_{h \to 0} (1 - h)$$

$= 1$ [put $h = 0$] **(1)**

$$\text{RHL} = \lim_{x \to \frac{1}{2}^+} f(x) = \lim_{x \to \frac{1}{2}^+} \left(\frac{3}{2} + x\right)$$

$$\text{RHL} = \lim_{h \to 0} \left(\frac{3}{2} + \frac{1}{2} + h\right)$$

$\left[\text{put } x = \frac{1}{2} + h, \text{ when } x \to \frac{1}{2}, h \to 0\right]$

$= \lim_{h \to 0} (2 + h) = 2$ [put $h = 0$] **(1)**

Now, we know that a function $f(x)$ is said to be continuous at point $x = a$, if

$$\text{LHL} = \text{RHL} = f(a).$$

Here, LHL $\neq$ RHL at $x = 1/2$.

Hence, $f(x)$ is discontinuous at $x = \frac{1}{2}$. **(1)**

**12.** Find the value of $a$, if the function $f(x)$ defined by

$$f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$$

is continuous at $x = 2$.

**All India 2011C; Delhi 2009C**

Given, $f(x) = \begin{cases} 2x - 1, & x < 2 \\ a, & x = 2 \\ x + 1, & x > 2 \end{cases}$

is continuous at point $x = 2$.

$\therefore$ By definition of continuity of a function at a point, we have

$$\text{LHL} = \text{RHL} = f(2) \quad \text{...(i)} \; \textbf{(1)}$$

Now, $\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1)$

$\Rightarrow \text{LHL} = \lim_{h \to 0} [2(2 - h) - 1]$

[put $x = 2 - h$, when $x \to 2, h \to 0$]

$= \lim_{h \to 0} (4 - 2h - 1) = \lim_{h \to 0} (3 - 2h)$

$= 3$ [put $h = 0$] **(1½)**

Also, from the given function, we have

$$f(2) = a \quad \textbf{(1/2)}$$

On putting the values of $f(2)$ and LHL in Eq. (i), we get

$$3 = a$$

$\Rightarrow$ $a = 3$ **(1)**

**13.** Find the values of $a$ and $b$ such that the function defined as follows is continuous.

$$f(x) = \begin{cases} x + 2, & x \le 2 \\ ax + b, & 2 < x < 5 \\ 3x - 2, & x \ge 5 \end{cases}$$

**Delhi 2010, 2010C**

- Given function is $f(x) = \begin{cases} x+2, & x \le 2 \\ ax+b, & 2 < x < 5 \\ 3x-2, & x \ge 5 \end{cases}$

Also, given that $f(x)$ is continuous at $x = 2$ and $x = 5$.

$\therefore$ By definition of continuity, we get

$$\text{LHL} = \text{RHL} = f(2) \quad \text{...(i)}$$

and $$\text{LHL} = \text{RHL} = f(5) \quad \text{...(ii)}$$

First, we find LHL and RHL at $x = 2$. **(1)**

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x+2)$$

$$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} (2 - h + 2)$$

[put $x = 2 - h$, when $x \to 2, h \to 0$]

$$= \lim_{h \to 0} (4 - h)$$

$$\Rightarrow \quad \text{LHL} = 4 \quad \text{[put } h = 0\text{]}$$

Now, $\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (ax + b)$

$$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [a(2+h) + b]$$

[put $x = 2 + h$, when $x \to 2, h \to 0$]

$$= \lim_{h \to 0} (2a + ah + b)$$

$$\Rightarrow \quad \text{RHL} = 2a + b \quad \text{[put } h = 0\text{]}$$

From Eq. (i), we have

$$\text{LHL} = \text{RHL}$$

$\therefore \quad 2a + b = 4 \qquad$ ...(iii) **(1)**

Now, we find LHL and RHL at $x = 5$.

$$\text{LHL} = \lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (ax + b)$$

$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} [a(5 - h) + b]$

[put $x = 5 - h$, when $x \to 5, h \to 0$]

$$= \lim_{h \to 0} (5a - ah + b)$$

$\Rightarrow \quad \text{LHL} = 5a + b \qquad$ [put $h = 0$]

and $\quad \text{RHL} = \lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (3x - 2)$

$$= \lim_{h \to 0} [3(5 + h) - 2]$$

[put $x = 5 + h$, when $x \to 5, h \to 0$]

$$= \lim_{h \to 0} (15 + 3h - 2)$$

$\Rightarrow \text{RHL} = 13 \qquad$ [put $h = 0$]

$\therefore$ From Eq. (ii), we have

$$\text{LHL} = \text{RHL}$$

$\Rightarrow \quad 5a + b = 13 \qquad$ ...(iv) **(1)**

On subtracting Eq. (iv) from Eq. (iii), we get

$$-3a = -9 \quad \Rightarrow a = 3$$

Put $a = 3$ in Eq. (iv), we get

$$15 + b = 13 \quad \Rightarrow b = -2$$

Hence, $\quad a = 3$ and $b = -2 \qquad$ **(1)**

**14.** For what value of $k$, is the function defined by

$f(x) = \begin{cases} k(x^2 + 2), & \text{if } x \le 0 \\ 3x + 1, & \text{if } x > 0 \end{cases}$, continuous at $x = 0$?

Also, write whether the function is continuous at $x = 1$. **Delhi 2010, 2010C**

The given function is

$$f(x) = \begin{cases} k(x^2+2), & \text{if } x \le 0 \\ 3x+1, & \text{if } x > 0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore$ $\quad$ $\text{LHL} = \text{RHL} = f(0)$ $\quad$ ...(i) **(1/2)**

Now, $\quad$ $\text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} k(x^2+2)$

$\Rightarrow$ $\quad$ $\text{LHL} = \lim_{h\to 0} k(h^2+2)$

[put $x = 0 - h = -h$, when $x \to 0, h \to 0$]

$= \lim_{h\to 0} (kh^2 + 2k)$

$\Rightarrow$ $\quad$ $\text{LHL} = 2k$ $\quad$ [put $h = 0$] **(1/2)**

and $\quad$ $\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3x+1)$

$\Rightarrow$ $\quad$ $\text{RHL} = \lim_{h\to 0} (3h+1)$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$\Rightarrow$ $\quad$ $\text{RHL} = 1$ $\quad$ [put $h = 0$] **(1/2)**

Now, from Eq. (i), we have

$$\text{LHL} = \text{RHL} \Rightarrow 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2} \quad \textbf{(1/2)}$$

Now, we check the continuity of the given function $f(x)$ at $x = 1$.

$$\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x + 1)$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} [3(1 - h) + 1]$$

[put $x = 1 - h$, when $x \to 1, h \to 0$]

$$= \lim_{h \to 0} (3 - 3h + 1)$$

$$= 4 \quad \text{[put } h = 0] \ \textbf{(1/2)}$$

$$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x + 1)$$

$$\Rightarrow \text{RHL} = \lim_{h \to 0} [3(1 + h) + 1]$$

[put $x = 1 + h$, when $x \to 1, h \to 0$]

$$= \lim_{h \to 0} [3 + 3h + 1]$$

$$= 4 \quad \text{[put } h = 0] \ \textbf{(1/2)}$$

and $f(1) = 4$ from given function.

$\because$ At $x = 1$, LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$. **(1)**

**15.** Find all points of discontinuity of $f$, where $f$ is defined as follows:

$$f(x) = \begin{cases} |x| + 3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x + 2, & x \ge 3 \end{cases}$$

Delhi 2010

Firstly, verify continuity of the given function at $x=-3$ and $x=3$. Then, point at which the given function is discontinuous will be the point of discontinuity.

The given function is $f(x) = \begin{cases} |x|+3, & x \le -3 \\ -2x, & -3 < x < 3 \\ 6x+2, & x \ge 3 \end{cases}$

First, we verify continuity at $x = -3$ and then at $x = 3$.

**Continuity at** $x = -3$

$$\text{LHL} = \lim_{x \to (-3)^-} f(x) = \lim_{x \to (-3)^-} (|x|+3)$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} (|-3-h|+3)$$

[put $x = -3-h$, when $x \to -3$, $h \to 0$]

$$= |-3|+3 \qquad \text{[put } h = 0]$$

$$= 3+3 \qquad [\because |-x| = x, \ \forall\, x \in R]$$

$$= 6$$

**(1/2)**

$$\text{RHL} = \lim_{x \to (-3)^+} f(x) = \lim_{x \to (-3)^+} (-2x)$$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [-2(-3+h)]$

[put $x = -3 + h$, when $x \to -3, h \to 0$]

$= \lim_{h \to 0} (6 - 2h)$

$\Rightarrow \quad \text{RHL} = 6$ [put $h = 0$] **(1/2)**

Also, $f(-3)$ = value of $f(x)$ at $x = -3$

$= |-3| + 3$

$= 3 + 3 = 6$ [$\because |-x| = x, \forall\ x \in R$] **(1/2)**

$\because \quad \text{LHL} = \text{RHL} = f(-3)$

$\therefore$ $f(x)$ is continuous at $x = -3$. So, $x = -3$ is the point of continuity. **(1/2)**

**Continuity at $x = 3$**

$\text{LHL} = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -2x$

$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} -2(3-h)$

[put $x = 3 + h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (-6 + 2h)$

$\Rightarrow \quad \text{LHL} = -6$ [put $h = 0$] **(1/2)**

and $\quad \text{RHL} = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2)$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [6(3+h) + 2]$

[put $x = 3 - h$, when $x \to 3, h \to 0$]

$= \lim_{h \to 0} (18 + 6h + 2)$

$\Rightarrow \quad \text{RHL} = 20$ [put $h = 0$] **(1/2)**

$\because \quad \text{LHL} \neq \text{RHL}$

As $f(x)$ is a polynomial function in a given interval, so it is continuous in a given interval but $f(x)$ is not continuous at $x = 3$. So, $x = 3$ is the point of discontinuity of $f(x)$. **(1)**

**16.** Show that the function $f(x)$ defined by

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x\ , & x>0\\ 2\ , & x=0\\ \dfrac{4(1-\sqrt{1-x})}{x}\ , & x<0\end{cases}$$

is continuous at $x=0$. **All India 2009C**

To show that the given function is continuous at $x=0$, we show that

$$(\text{LHL})_{x=0}=(\text{RHL})_{x=0}=f(0) \qquad \ldots\text{(i)}\ \textbf{(1)}$$

Now, given function is

$$f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & x>0\\ 2, & x=0\\ \dfrac{4(1-\sqrt{1-x})}{x}, & x<0\end{cases}$$

$$\text{LHL}=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{4(1-\sqrt{1-x})}{x}$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} \frac{4[1-\sqrt{1-(0-h)}]}{0-h}$$

[put $x = 0 - h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1+h}]}{-h}$$

$$= \lim_{h \to 0} \frac{4[1-\sqrt{1+h}]}{-h} \times \frac{1+\sqrt{1+h}}{1+\sqrt{1+h}}$$

$$= \lim_{h \to 0} \frac{4[(1)^2-(\sqrt{1+h})^2]}{-h[1+\sqrt{1+h}]}$$

[$\because (a-b)(a+b) = a^2 - b^2$]

$$= \lim_{h \to 0} \frac{4[1-(1+h)]}{-h[1+\sqrt{1+h}]}$$

$$= \lim_{h \to 0} \frac{-h \times 4}{-h[1+\sqrt{1+h}]} = \lim_{h \to 0} \frac{4}{1+\sqrt{1+h}}$$

$$= \frac{4}{1+\sqrt{1}} = \frac{4}{2} = 2 \qquad \text{[put } h = 0]$$

$\Rightarrow \text{LHL} = 2$ **(1)**

Now, $\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{\sin x}{x} + \cos x\right)$

$$\Rightarrow \text{RHL} = \lim_{h \to 0} \left(\frac{\sin h}{h} + \cos h\right)$$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{\sin h}{h} + \lim_{h \to 0} \cos h \qquad \text{[put } h = 0]$$

$$= 1 + \cos 0^\circ \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

$$= 1 + 1 \qquad [\because \cos 0 = 1]$$

$= 2$ **(1)**

Also, given that at $x = 0$, $f(x) = 2 \Rightarrow f(0) = 2$

Since, $(\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0) = 2$

Hence, $f(x)$ is continuous at $x = 0$. **(1)**

**17.** For what value of $k$, is the following function continuous at $x = 2$?

$$f(x) = \begin{cases} 2x+1 & , x<2 \\ k & , x=2 \\ 3x-1 & , x>2 \end{cases}$$

**Delhi 2008**

Do same as Que 12. **[Ans.** $k = 5$**]**

**18.** If $f(x)$ defined by the following, is continuous at $x = 0$, then find the values of $a, b$ and $c$.

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x<0 \\ c & , \text{if } x=0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , \text{if } x>0 \end{cases}$$

**HOTS; All India 2008**

Given function is

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x<0 \\ c & , \text{if } x=0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , \text{if } x>0 \end{cases}$$

Also, given that $f(x)$ is continuous at $x = 0$.

$\therefore \quad (\text{LHL})_{x=0} = (\text{RHL})_{x=0} = f(0)$ ...(i)

Now, $\quad \text{LHL} = \lim_{x \to 0^-} f(x)$

$$= \lim_{x \to 0^-} \frac{\sin(a+1)x + \sin x}{x}$$

$$\Rightarrow \text{LHL} = \lim_{h \to 0} \frac{\sin[-(a+1)h] + \sin(-h)}{-h}$$

[put $x = 0 - h = -h$, when $x \to 0, h \to 0$]

$$= \lim_{h \to 0} \frac{-\sin(a+1)h - \sin h}{-h}$$

[$\because \sin(-\theta) = -\sin\theta$]

$$= \lim_{h \to 0} \frac{\sin(a+1)h + \sin h}{h}$$

$$2\sin\left[\frac{(a+1)h+h}{2}\right]\cos\left[\frac{(a+1)h-h}{2}\right]$$

$$= \lim_{h\to 0} \frac{\left[ \quad 2 \quad \right] \left[ \quad 2 \quad \right]}{h}$$

$$\left[\because \sin C + \sin D = 2 \sin \frac{(C+D)}{2} \cdot \cos \frac{(C-D)}{2}\right]$$

$$= \lim_{h\to 0} \frac{2 \sin\left[\frac{(a+1)h+h}{2}\right] \cos\left[\frac{(a+1)h-h}{2}\right]}{\left[\frac{(a+1)h+h}{2}\right] \times h} \times \frac{(a+1)h+h}{2}$$

$$\left[\text{multiplying and dividing by } \frac{(a+1)h+h}{2}\right]$$

$$= \lim_{h\to 0} \frac{2 \cos\left[\frac{(a+1)h-h}{2}\right]}{h} \times \frac{h[a+1+1]}{2}$$

$$\left[\begin{array}{l} \because \lim_{x\to 0} \frac{\sin x}{x} = 1 \\ \therefore \lim_{h\to 0} \frac{\frac{\sin[(a+1)h+h]}{2}}{\frac{(a+1)h+h}{2}} = 1 \end{array}\right]$$

$$= \lim_{h\to 0} \frac{2(a+2)}{2} \cos\left[\frac{(a+1)h-h}{2}\right]$$

$= (a+2) \cos 0°$ [put $h = 0$]

$= (a+2) \times 1$ [$\because \cos 0° = 1$]

$= a + 2$ **(1½)**

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}$$

$$\Rightarrow \quad \text{RHL} = \lim_{h\to 0} \frac{\sqrt{h+bh^2} - \sqrt{h}}{bh^{3/2}}$$

[put $x = 0 + h = h$, when $x \to 0, h \to 0$]

$$= \lim_{h\to 0} \frac{\sqrt{h(1+bh)} - \sqrt{h}}{bh^{3/2}}$$

$$= \lim_{h \to 0} \frac{\sqrt{h}\,[\sqrt{1+bh} - 1]}{b\sqrt{h} \cdot h} [\because\ h^{3/2} = \sqrt{h} \cdot h]$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{bh}$$

$$= \lim_{h \to 0} \frac{\sqrt{1+bh} - 1}{bh} \times \frac{\sqrt{1+bh} + 1}{\sqrt{1+bh} + 1}$$

[multiplying and dividing by $\sqrt{1+bh} + 1$]

$$= \lim_{h \to 0} \frac{(\sqrt{1+bh})^2 - (1)^2}{bh\,[\sqrt{1+bh} + 1]}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \lim_{h \to 0} \frac{1 + bh - 1}{bh\,[\sqrt{1+bh} + 1]}$$

$$= \lim_{h \to 0} \frac{bh}{bh\,[\sqrt{1+bh} + 1]}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+bh} + 1}$$ **(1½)**

$\Rightarrow$ RHL $= \dfrac{1}{1+1} = \dfrac{1}{2}$ [put $h = 0$]

Now, from Eq. (i), we have

LHL = RHL

$\Rightarrow \quad a + 2 = \dfrac{1}{2}$

$\Rightarrow \quad a = \dfrac{1}{2} - 2 \Rightarrow a = -\dfrac{3}{2}$

Also, given that $f(0) = c$

Again from Eq. (i), we have

$\text{RHL} = f(0)$

$\Rightarrow \quad c = 1/2$

Hence, we get $a = -\dfrac{3}{2}$, $c = \dfrac{1}{2}$ and $b$ may take any real value. **(1)**

**NOTE** Here, we cannot find any real and unique value of $b$ that means $b$ may take any real value, i.e. $b \in R$.

**19.** Show that $f(x) = \begin{cases} 5x-4, & \text{when } 0 < x \le 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$

is continuous at $x = 1$. **Delhi 2008C**

Given function is

$$f(x) = \begin{cases} 5x-4, & \text{when } 0 < x \le 1 \\ 4x^3 - 3x, & \text{when } 1 < x < 2 \end{cases}$$

To show that $f(x)$ is continuous at $x = 1$, we need to prove

$$\text{LHL}_{x=1} = \text{RHL}_{x=1} = f(1) \quad \ldots\text{(i)} \quad \textbf{(1)}$$

Now, $\text{LHL} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5x-4)$

$\Rightarrow \quad \text{LHL} = \lim_{h \to 0} [5(1-h) - 4]$

[put $x = 1-h$, when $x \to 1$, $h \to 0$]

$= \lim_{h \to 0} (5 - 5h - 4) = \lim_{h \to 0} (1 - 5h)$

$\text{LHL} = 1$ [put $h = 0$] **(1)**

$\text{RHL} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x^3 - 3x)$

$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [4(1+h)^3 - 3(1+h)]$

[put $x = 1+h$, when $x \to 1$, $h \to 0$]

$= 4(1)^3 - 3(1)$ [put $h = 0$]

$\Rightarrow \quad \text{RHL} = 4 - 3 = 1$ **(1)**

Also, from given function,

$f(1) =$ Value of $f(x)$ at $x = 1$

$\Rightarrow \quad f(1) = 5(1) - 4$

[put $x = 1$ in $f(x) = 5x - 4$]

$= 5 - 4 = 1$

Here, $(\text{LHL})_{x=1} = (\text{RHL})_{x=1} = f(1)$ **(1)**

Hence, $f(1)$ is continuous at $x = 1$.

**20.** If the following function $f(x)$ is continuous at $x = 0$, then find the value of $k$.

$$f(x) = \begin{cases} \dfrac{1 - \cos 2x}{2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}.$$

All India 2008C

Do same as Que 1. [Ans. $k = 1$]

## Differntiability

### 1 Mark Questions

**1.** Write the derivative of $\sin x$ with respect to $\cos x$. **Delhi 2014C**

Let $u = \sin x$

On differentiating $u$ w.r.t. $x$, we get

$$\frac{du}{dx} = \cos x \quad \ldots(i)$$

and $v = \cos x$

On differentiating $v$ w.r.t. $x$, we get

$$\frac{dv}{dx} = -\sin x \quad \ldots(ii)$$

Now, $\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$

$= -\frac{\cos x}{\sin x}$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad \frac{du}{dv} = -\cot x$ **(1)**

**2.** If $\cos y = x \cos(a + y)$, where $\cos a \neq \pm 1$, prove that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$. **Foreign 2014**

Given, $\cos y = x \cos(a + y)$

$$\Rightarrow \qquad x = \frac{\cos y}{\cos(a+y)}$$

On differentiating both sides w.r.t. $y$, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\frac{d}{dy}(\cos y) - \cos y \frac{d}{dy}[\cos(a+y)]}{\cos^2(a+y)}$$

[by using quotient rule]

$$\Rightarrow \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y[-\sin(a+y)]}{\cos^2(a+y)}$$

$$= \frac{\cos y \sin(a+y) - \cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

[$\because \sin A \cos B - \cos A \sin B = \sin(A - B)$]

$$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \qquad (1)$$

**Hence proved.**

**3.** If $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$.

**All India 2014C; Delhi 2010**

> Firstly, convert the given expression in $\sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$ form and then put $x = \sin\phi$ and $y = \sin\theta$. Now, simplify the resulting expression and differentiate it.

Given, $y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$

Above equation can be rewritten as

$$y = \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$$[\because \; x = (\sqrt{x})^2]$$

Now, put $\sqrt{x} = \sin\theta$ and $x = \sin\phi$, so that

$\theta = \sin^{-1}\sqrt{x}$ and $\phi = \sin^{-1} x$, we get

$$y = \sin^{-1}[\sin\phi\sqrt{1-\sin^2\theta} - \sin\theta\sqrt{1-\sin^2\phi}]$$

$$\Rightarrow \quad y = \sin^{-1}[\sin\phi\cos\theta - \sin\theta\cos\phi]$$

$$[\because \; \sqrt{1-\sin^2 x} = \cos x]$$

$$\Rightarrow \quad y = \sin^{-1}\sin(\phi - \theta)$$

$$[\because \; \sin\phi\cos\theta - \sin\theta\cos\phi = \sin(\phi - \theta)]$$

$$\Rightarrow \quad y = \phi - \theta \qquad [\because \sin^{-1}\sin\theta = \theta]$$

$$\Rightarrow \quad y = \sin^{-1} x - \sin^{-1}\sqrt{x}$$

$$[\because \; \phi = \sin^{-1} x \text{ and } \theta = \sin^{-1}\sqrt{x}]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}(\sqrt{x})$$

$$\left[\because \frac{d}{d\theta}(\sin^{-1}\theta) = \frac{1}{\sqrt{1-\theta^2}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\left[\because \; \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\right]$$

Hence, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{x-x^2}}$ **(1)**

***Alternate Method***

> Use the formula,
>
> $\sin^{-1} x - \sin^{-1} y = \sin^{-1}[x\sqrt{1-y^2} - y\sqrt{1-x^2}]$
>
> and then differentiate with respect to $x$ to get the required value.

$$\because \quad y = \sin^{-1}[x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$$

$$\Rightarrow \quad y = \sin^{-1}[x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}]$$

$$[\because \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\} = \sin^{-1}x - \sin^{-1}y]$$

Here, $x = x$ and $y = \sqrt{x}$

$$\therefore \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x) - \frac{d}{dx}(\sin^{-1}\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$$ **(1)**

**4.** If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx} + e^{y-x} = 0$.

**Foregin 2014**

Given, $e^x + e^y = e^{x+y}$ ...(i)

On dividing Eq. (i) by $e^{x+y}$, we get

$$e^{-y} + e^{-x} = 1 \quad \text{...(ii)}$$

On differentiating both sides of Eq. (ii) w.r.t. $x$, we get

$$e^{-y} \cdot \left(\frac{-dy}{dx}\right) + e^{-x}(-1) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-e^{-x}}{e^{-y}} = -e^{(y-x)}$$

$$\Rightarrow \quad \frac{dy}{dx} + e^{(y-x)} = 0$$ **(1)**

**5.** Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$.

**All India 2014**

Given, $x = ae^{\theta}(\sin\theta - \cos\theta)$

On differentiating w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}[e^{\theta}\sin\theta - e^{\theta}\cos\theta]$$

$$= a\left[\frac{d}{d\theta}(e^{\theta}\sin\theta) - \frac{d}{d\theta}(e^{\theta}\cos\theta)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^{\theta}) - e^{\theta}\frac{d}{d\theta}(\cos\theta) - \cos\theta\frac{d}{d\theta}(e^{\theta})\right]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta - e^{\theta}(-\sin\theta) - e^{\theta}\cos\theta]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta + e^{\theta}\sin\theta - e^{\theta}\cos\theta]$$

$$\Rightarrow \frac{dx}{d\theta} = a[2e^{\theta}\sin\theta] = 2ae^{\theta}\sin\theta \quad \text{...(i)}$$

Also, $y = ae^{\theta}(\sin\theta + \cos\theta)$

On differentiating w.r.t. $\theta$, we get

$$\frac{dy}{d\theta} = a\left[\frac{d}{d\theta}(e^{\theta}\sin\theta) + \frac{d}{d\theta}(e^{\theta}\cos\theta)\right]$$

$$= a\left[e^{\theta}\frac{d}{d\theta}(\sin\theta) + \sin\theta\frac{d}{d\theta}(e^{\theta}) + e^{\theta}\frac{d}{d\theta}(\cos\theta) + \cos\theta\frac{d}{d\theta}(e^{\theta})\right]$$

$$= a[e^{\theta}\cos\theta + e^{\theta}\sin\theta - e^{\theta}\sin\theta + e^{\theta}\cos\theta]$$

$$= a[2e^{\theta}\cos\theta]$$

$$\Rightarrow \quad \frac{dy}{d\theta} = 2ae^{\theta}\cos\theta \quad \text{...(ii)}$$

Now, $$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$$

[from Eqs. (i) and (ii)]

$$= \cot\theta$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{dx} = \cot\frac{\pi}{4}$

Hence, $\frac{dy}{dx} = 1$ $\quad \left[\because \cot\frac{\pi}{4} = 1\right]$ **(1)**

**6.** If $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$, then evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. **Delhi 2014C**

Given, $x = a\left(\cos t + \log \tan\frac{t}{2}\right)$

and $y = a \sin t$. ...(i)

Now, $x = a\left(\cos t + \log \tan\frac{t}{2}\right)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dx}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\log \tan\frac{t}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}}\frac{d}{dt}\left(\tan\frac{t}{2}\right)\right]$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \cdot \sec^2\frac{t}{2}\frac{d}{dt}\left(\frac{t}{2}\right)\right]$$

$$= a\left[-\sin t + \frac{1}{\tan\frac{t}{2}} \times \sec^2\frac{t}{2} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{\frac{\sin t/2}{\cos t/2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2\sin\frac{t}{2} \cdot \cos\frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] \quad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right) \Rightarrow \frac{dx}{dt} = a\left(\frac{\cos^2 t}{\sin t}\right) \quad ...(ii)$$

$$[\because 1-\sin^2\theta = \cos^2\theta]$$

and $\quad y = a \sin t$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = a \cos t \quad \text{...(iii)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t}{a\left(\frac{\cos^2 t}{\sin t}\right)}$$

[from Eqs. (ii) and (iii)]

$$= \frac{a \cos t}{a \cos^2 t} \times \sin t = \tan t$$

On differentiating both sides of above equation w.r.t.x, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t)$$

$$= \frac{d}{dt}(\tan t)\frac{dt}{dx} \quad \left[\because \frac{d}{dx} f(t) = \frac{d}{dt} f(t) \cdot \frac{dt}{dx}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \times \frac{\sin t}{a \cos^2 t}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\sin t \sec^4 t}{a}$$

Now, on putting $t = \frac{\pi}{3}$, we get

$$\left[\frac{d^2y}{dx^2}\right]_{t=\frac{\pi}{3}} = \frac{\sin\frac{\pi}{3} \times \sec^4\frac{\pi}{3}}{a} = \frac{\frac{\sqrt{3}}{2} \times (2)^4}{a}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\frac{\sqrt{3}}{2} \times 16}{a}$$

Hence, $\quad \frac{d^2y}{dx^2} = \frac{8\sqrt{3}}{a}$ **(1)**

7. If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

**Foreign 2014**

Given, $x^m y^n = (x+y)^{m+n}$

On taking log both sides, we get

$$\log(x^m y^n) = \log(x+y)^{m+n}$$

$$\Rightarrow \quad \log(x^m) + \log(y^n) = (m+n)\log(x+y)$$

$$\Rightarrow \quad m\log x + n\log y = (m+n)\log(x+y)$$

On differentiating w.r.t. x, we get

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{m}{x} - \frac{(m+n)}{x+y} = \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\frac{dy}{dx}$$

$$\Rightarrow \left[\frac{my+ny-nx-ny}{y(x+y)}\right]\frac{dy}{dx} = \frac{mx+my-mx-nx}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[\frac{my-nx}{y}\right] = \frac{my-nx}{x}$$

Hence, $$\frac{dy}{dx} = \frac{y}{x}$$ **(1)**

8. If $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$.

**Foreign 2014**

Given, $x = a\cos\theta + b\sin\theta$ ...(i)

and $y = a\sin\theta - b\cos\theta$ ...(ii)

On differentiating Eq. (i) w.r.t. θ, we get

$$\frac{dx}{d\theta} = -a\sin\theta + b\cos\theta$$

On differentiating Eq. (ii) w.r.t. θ, we get

$$\frac{dy}{d\theta} = a\cos\theta + b\sin\theta$$

Now, $$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{-y} \quad \left[\because \begin{array}{l} x = a\cos\theta + b\sin\theta, \\ y = a\sin\theta - b\cos\theta \end{array}\right]$$

$$\Rightarrow y\frac{dy}{dx} = -x \Rightarrow y\frac{dy}{dx} + x = 0 \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. $x$, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

$$\Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{-x}{y}\right) + 1 = 0 \quad \left[\because \frac{dy}{dx} = \frac{-x}{y}\right]$$

$$\Rightarrow y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad \textbf{(1)}$$

**Hence proved.**

**9.** Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ w.r.t. $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **Delhi 2014**

Let $u = \tan^{-1}\left[\frac{\sqrt{1-x^2}}{x}\right]$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

Then, $u = \tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]$

$$\left[\quad \cos\theta \quad\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\sin^2\theta}}{\cos\theta}\right] = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right]$$

$$= \tan^{-1}[\tan\theta] = \theta$$

$\Rightarrow \qquad u = \cos^{-1} x \qquad [\because x = \cos\theta]$

On differentiating w.r.t. x, we get

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Again, let $v = \cos^{-1}(2x\sqrt{1-x^2})$

Put $\qquad x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

Then, $\qquad v = \cos^{-1}[2\cos\theta\sqrt{1-\cos^2\theta}]$

$$= \cos^{-1}[2\cos\theta\sin\theta]$$

$$\left[\begin{array}{l}\because \cos^2\theta + \sin^2\theta = 1 \\ \sin\theta = \sqrt{1-\cos^2\theta}\end{array}\right]$$

$$= \cos^{-1}[\sin 2\theta]$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2\theta\right)\right] = \frac{\pi}{2} - 2\theta$$

$\Rightarrow \qquad v = \frac{\pi}{2} - 2\cos^{-1} x$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$$

Now, $\qquad \frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$

$$= -\frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2} = -\frac{1}{2} \qquad (1)$$

**10.** If $y = x^x$, then prove that

$$\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$

**Delhi 2014**

Given, $y = x^x$

On taking log both sides, we get

$$\log y = \log x^x$$

$$\Rightarrow \quad \log y = x \log x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (1 + \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1 + \log x) \quad \ldots(i)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y\frac{d}{dx}(1 + \log x) + (1 + \log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = y \times \frac{1}{x} + (1 + \log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + (1 + \log x)\frac{dy}{dx} \quad \ldots(ii)$$

Now, we have to prove that

$$\frac{d^2y}{dx^2} - \frac{-1}{y}\left(\frac{dy}{dx}\right)^2 \frac{-y}{x} = 0$$

$$\text{LHS} = \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}$$

$$= \frac{y}{x} + (1 + \log x)\frac{dy}{dx} - \frac{1}{y}[y(1 + \log x)]^2 - \frac{y}{x}$$

[from Eqs. (i) and (ii)]

$$= \frac{y}{x} + (1 + \log x)\, y(1 + \log x) - \frac{1}{y}[y^2(1 + \log x)^2] - \frac{y}{x} \quad \text{[from Eq. (i)]}$$

$$= y(1 + \log x)^2 - y(1 + \log x)^2 = 0 = \text{RHS}$$

**Hence proved. (1)**

**11.** Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ w.r.t. $\sin^{-1}(2x\sqrt{1-x^2})$. **Delhi 2014**

Let $u = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, then

$$u = \tan^{-1}\left[\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right]$$

$$\Rightarrow \quad u = \tan^{-1}\left[\frac{\sin\theta}{\cos\theta}\right] \quad \left[\begin{array}{l}\because \sin^2\theta + \cos^2\theta = 1 \\ \Rightarrow \cos\theta = \sqrt{(1-\sin^2\theta)}\end{array}\right]$$

$$\Rightarrow \quad u = \tan^{-1}(\tan\theta) \Rightarrow u = \theta \quad \Rightarrow u = \sin^{-1} x$$

On differentiating w.r.t. $x$, we get

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \quad ...(i)$$

Again, let $v = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$, then

$$v = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$$

$$\Rightarrow \quad v = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow \quad v = \sin^{-1}(\sin 2\theta) \Rightarrow v = 2\theta$$

$$\Rightarrow \quad v = 2\sin^{-1}x$$

On differentiating w.r.t. $x$, we get

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \quad ...(ii)$$

Now, $$\frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv}$$

$$\Rightarrow \quad \frac{du}{dv} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{2}$$

$$\therefore \quad \frac{du}{dv} = \frac{1}{2}$$ **(1)**

**12.** Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$.

**Delhi 2014**

Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \cot\theta \Rightarrow \theta = \cot^{-1} x$, then

$$u = \tan^{-1}\left[\frac{\sqrt{1+\cot^2\theta}-1}{\cot\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{\operatorname{cosec}^2\theta}-1}{\cot\theta}\right]$$

$$= \tan^{-1}\left[\frac{\operatorname{cosec}\theta-1}{\cot\theta}\right] = \tan^{-1}\left[\frac{1-\sin\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right]$$

$[\because \sin^2 x + \cos^2 x = 1]$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}\right]$$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$$= \tan^{-1}\left[\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right] = \tan^{-1}\left[\frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\right]$$

[$\because$ dividing numerator and denominator by $\cos\theta/2$]

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\Rightarrow \quad u = \frac{\pi}{4} - \frac{\theta}{2} = \frac{\pi}{4} - \frac{\cot^{-1}x}{2}$$

On differentiating w.r.t. $x$, we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)} \qquad \text{...(i)}$$

$$\left[\because \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}\right]$$

Again, let $\quad v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $\quad x = \tan\theta \Rightarrow \theta = \tan^{-1}x$, then we get

$$v = \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$\Rightarrow v = \sin^{-1}[\sin 2\theta] \Rightarrow v = 2\theta \Rightarrow v = 2\tan^{-1}x$$

On differentiating w.r.t. $x$, we get

$$\frac{dv}{dx} = \frac{2}{1+x^2} \qquad \text{...(ii)}$$

Now, $\quad \frac{du}{dv} = \frac{du}{dx} \times \frac{dx}{dv} = \frac{1}{2(1+x^2)} \times \frac{(1+x^2)}{2}$

[from Eqs. (i) and (ii)]

$$\Rightarrow \quad \frac{du}{dv} = \frac{1}{4} \qquad \textbf{(1)}$$

**13.** If $y = Pe^{ax} + Qe^{bx}$, then show that

$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0.$ **All India 2014, 2009C**

Given, $y = Pe^{ax} + Qe^{bx}$ ...(i)

On differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = P\frac{d}{dx}(e^{ax}) + Q\frac{d}{dx}(e^{bx})$$

$$\Rightarrow \quad \frac{dy}{dx} = Pa\, e^{ax} + Qb\, e^{bx} \quad \text{...(ii)}$$

Again, differentiating w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = Pa\frac{d}{dx}(e^{ax}) + bQ\frac{d}{dx}(e^{bx})$$

$$= Pa\,(a\, e^{ax}) + bQ\,(b\, e^{bx})$$

$$= a^2P\, e^{ax} + b^2Q\, e^{bx} \quad \text{...(iii)}$$

Now, LHS $= \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$

On putting values from Eqs. (i), (ii) and (iii), we get

LHS $= a^2P\, e^{ax} + b^2Q\, e^{bx}$

$- (a+b)(aP\, e^{ax} + bQ\, e^{bx}) + ab\,(P\, e^{ax} + Q\, e^{bx})$

$= a^2P\, e^{ax} + b^2Q\, e^{bx} - a^2P\, e^{ax} - ab\, Q\, e^{bx}$

$\quad - ab\, P\, e^{ax} - b^2Q\, e^{bx} + ab\, P\, e^{ax} + ab\, Q\, e^{bx}$

$= 0 =$ RHS **(1)**

**Hence proved.**

**14.** If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, then find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. **All India 2014**

Given, $x = \cos t\,(3 - 2\cos^2 t)$

$\Rightarrow \quad x = 3\cos t - 2\cos^3 t$

On differentiating w.r.t. $t$, we get

$$\frac{dx}{dt} = 3(-\sin t) - 2(3)\cos^2 t\,(-\sin t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -3\sin t + 6\cos^2 t \sin t \qquad \text{...(i)}$$

Also, $y = \sin t\,(3 - 2\sin^2 t)$

$\Rightarrow \quad y = 3\sin t - 2\sin^3 t$

On differentiating w.r.t. $t$, we get

$$\frac{dy}{dt} = 3\cos t - 2 \times 3 \times \sin^2 t \cos t$$

$$\Rightarrow \quad \frac{dy}{dt} = 3\cos t - 6\sin^2 t \cos t \qquad \text{...(ii)}$$

Now, $$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3\cos t - 6\cos t \sin^2 t}{-3\sin t + 6\cos^2 t \sin t}$$

[from Eqs. (i) and (ii)]

$$= \frac{\cos t - 2\cos t \sin^2 t}{-\sin t + 2\cos^2 t \sin t}$$

At $t = \frac{\pi}{4}$,

$$\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{\cos\frac{\pi}{4} - 2\cos\frac{\pi}{4}\sin^2\frac{\pi}{4}}{-\sin\frac{\pi}{4} + 2\cos^2\frac{\pi}{4}\sin\frac{\pi}{4}}$$

$$= \frac{\frac{1}{\sqrt{2}} - 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^2}{-\frac{1}{\sqrt{2}} + 2\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)} = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = 0$$

**15.** If $(x - y)\,e^{\frac{x}{x-y}} = a$. Prove that $y\frac{dy}{dx} + x = 2y$.

**Delhi 2014C**

Given, $(x-y)\cdot e^{\frac{x}{x-y}} = a$

On taking log both sides, we get

$$\log\left[(x-y)\cdot e^{\frac{x}{x-y}}\right] = \log a$$

$$\Rightarrow \quad \log(x-y) + \log e^{\frac{x}{x-y}} = \log a$$

$$[\because \log(mn) = \log m + \log n]$$

$$\Rightarrow \quad \log(x-y) + \frac{x}{x-y}\log_e e = \log a$$

$$\Rightarrow \quad \log(x-y) + \frac{x}{x-y} = \log a$$

$$[\because \log_e = 1]$$

On differentiating w.r.t. x, we get

$$\frac{d}{dx}[\log(x-y)] + \frac{d}{dx}\left(\frac{x}{x-y}\right) = \frac{d}{dx}(\log a)$$

$$\Rightarrow \frac{1}{x-y}\frac{d}{dx}(x-y)$$

$$+\frac{(x-y)\frac{d}{dx}(x) - x\frac{d}{dx}(x-y)}{(x-y)^2} = 0$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u\frac{d}{dx}(v) - v\frac{d}{dx}(u)}{v^2}\right]$$

$$\Rightarrow \quad \frac{1}{x-y}\cdot(1-y') + \frac{(x-y) - x(1-y')}{(x-y)^2} = 0$$

where, $y' = dy/dx$

$$\Rightarrow \quad (x-y)(1-y') + x - y - x(1-y') = 0$$

$$\Rightarrow \quad yy' + x - 2y = 0$$

$$\Rightarrow \quad y\frac{dy}{dx} + x = 2y \qquad \textbf{(1)}$$

**Hence proved.**

**16.** If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. **Delhi 2014C**

Given, $x = a(\cos t + t \sin t)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dx}{dt} = a(-\sin t + 1 \cdot \sin t + t \cos t)$$

$$\Rightarrow \quad \frac{dx}{dt} = a\, t \cos t$$

and $\quad y = a(\sin t - t \cos t)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = a(\cos t - \cos t \cdot 1 + t \sin t) = at \sin t \quad \text{...(i)}$$

Now, $\quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{at \sin t}{at \cos t} = \tan t$

Again differentiating both sides w.r.t. $x$, we get

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(\tan t)\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{1}{\frac{dx}{dt}} = \frac{\sec^2 t}{at \cos t} = \frac{\sec^3 t}{at}$$

At $t = \frac{\pi}{4}$,

$$\therefore \left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = \frac{\sec^3\left(\frac{\pi}{4}\right)}{a\left(\frac{\pi}{4}\right)} = \frac{(\sqrt{2})^3 \cdot 4}{2\pi}$$

$$= \frac{8\sqrt{2}}{a\pi} \quad \textbf{(1)}$$

**17.** If $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx} = \frac{2a^3}{x^4 - a^4}$.

**All India 2014C**

Given, $y = \tan^{-1}\left(\frac{a}{x}\right) + \log\sqrt{\frac{x-a}{x+a}}$

$$= \tan^{-1}\left(\frac{a}{x}\right) + \log\left(\frac{x-a}{x+a}\right)^{1/2}$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{a}{x}\right) + \frac{1}{2}[\log(x-a) - \log(x+a)]$$

$$\left[\because \log\frac{m}{n} = \log m - \log n\right]$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+\frac{a^2}{x^2}} \cdot \left(\frac{-a}{x^2}\right) + \frac{1}{2}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\left[\because \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \text{ and } \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= \frac{-a}{x^2+a^2} + \frac{a}{x^2-a^2}$$

$$= \frac{-x^2a + a^3 + x^2a + a^3}{x^4 - a^4}$$

$$[\because (a+b)(a-b) = a^2 - b^2]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a^3}{x^4 - a^4} \qquad \textbf{(1)}$$

**Hence proved.**

**18.** If $x = a\sin 2t\,(1+\cos 2t)$ and $y = b\cos 2t\,(1-\cos 2t)$, then show that at $t = \frac{\pi}{4}, \left(\frac{dy}{dx}\right) = \frac{b}{a}$.

**All India 2014**

Given, $x = a\sin 2t\,(1+\cos 2t)$

Given, $x = a \sin 2t\,(1 + \cos 2t)$

$$\Rightarrow \quad x = a \sin 2t\,(2\cos^2 t)$$

$$[\because \cos 2\theta = 2\cos^2\theta - 1]$$

$$\Rightarrow \quad x = 2a \sin 2t \cos^2 t$$

On differentiating $x$ w.r.t. $t$, we get

$$\frac{dx}{dt} = 2a\left[\sin 2t \frac{d}{dt}(\cos^2 t) + \cos^2 t \frac{d}{dt}(\sin 2t)\right]$$

$$= 2a\,[\sin 2t\,\{2\cos t\,(-\sin t)\} + 2\cos^2 t\,(\cos 2t)]$$

$$= 2a\,[-\sin^2 2t + 2\cos^2 t \cos 2t]$$

$$[\because 2\sin\theta\cos\theta = \sin 2\theta]$$

Also, $y = b\cos 2t\,(1 - \cos 2t)$

$$= b\cos 2t\,(2\sin^2 t)$$

$$[\because \cos 2\theta = 1 - 2\sin^2\theta]$$

$$= 2b\cos 2t \sin^2 t$$

On differentiating $y$ w.r.t. $t$, we get

$$\frac{dy}{dt} = 2b\left[\cos 2t \frac{d}{dt}(\sin^2 t) + \sin^2 t \frac{d}{dt}(\cos 2t)\right]$$

$$= 2b\,[\cos 2t\,(2\sin t\cos t) + \sin^2 t\,(-\sin 2t)\cdot 2]$$

$$= 2b\,[\cos 2t \sin 2t - 2\sin^2 t \sin 2t]$$

Now, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2b\,[\cos 2t\sin 2t - 2\sin^2 t\sin 2t]}{2a\,[2\cos^2 t\cos 2t - \sin^2 2t]}$$

At $t = \dfrac{\pi}{4}$,

$$\left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{b}{a}\,\frac{\left[\cos 2\left(\frac{\pi}{4}\right)\sin 2\left(\frac{\pi}{4}\right) - 2\sin^2\left(\frac{\pi}{4}\right)\sin 2\left(\frac{\pi}{4}\right)\right]}{\left[2\cos^2\left(\frac{\pi}{4}\right)\cos 2\left(\frac{\pi}{4}\right) - \sin^2 2\left(\frac{\pi}{4}\right)\right]}$$

$$= \frac{b}{a}\,\frac{\left[\cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - 2\times\left(\frac{1}{\sqrt{2}}\right)^2\sin\left(\frac{\pi}{2}\right)\right]}{\left[2\times\left(\frac{1}{\sqrt{2}}\right)^2\cos\left(\frac{\pi}{2}\right) - \sin^2\left(\frac{\pi}{2}\right)\right]}$$

$$\left[\because \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{b}{a}\frac{[0\times 1 - 1\times 1]}{[1\times 0 - 1]}$$

$$= \frac{b}{a}\left(\frac{-1}{-1}\right) \qquad \left[\because \sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0\right]$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{b}{a} \qquad \textbf{(1)}$$

**Hence proved.**

**19.** If $(\tan^{-1} x)^y + y^{\cot x} = 1$, then find $dy/dx$.

**All India 2014C**

Let $u = (\tan^{-1} x)^y$ and $v = y^{\cot x}$

Then, given equation becomes $u + v = 1$

On differentiating both sides w.r.t. $x$, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \qquad \ldots\text{(i)}$$

Now, $\quad u = (\tan^{-1} x)^y$

On taking log both sides, we get

$$\log u = y\log(\tan^{-1} x)$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{\tan^{-1} x\,(1+x^2)}$$

$$\Rightarrow \quad \frac{du}{dx} = (\tan^{-1} x)^y \left[\frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{\tan^{-1} x\,(1+x^2)}\right] \ldots\text{(ii)}$$

Also, $\quad v = y^{\cot x}$

On taking log both sides, we get

$$\log v = \cot x \log y$$

On differentialting w.r.t. $x$, we get

$$\frac{1}{v}\frac{dv}{dx} = -\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x}\left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dy}{dx}\right] \quad \ldots\text{(iii)}$$

On putting values from Eqs. (ii) and (iii) in Eq. (i), we get

$$(\tan^{-1} x)^y \left[\frac{dv}{dx}\log(\tan^{-1} x) + \frac{y}{\tan^{-1} x\,(1+x^2)}\right] + y^{\cot x}\left[-\operatorname{cosec}^2 x \log y + \frac{\cot x}{y}\frac{dv}{dx}\right] = 0$$

$$\Rightarrow \frac{dy}{dx}\left[(\tan^{-1} x)^y \log(\tan^{-1} x) + \cot x\, y^{\cot x - 1}\right]$$

$$= -\left[\frac{y}{1+x^2}(\tan^{-1} x)^{y-1} - y^{\cot x}\operatorname{cosec}^2 x \log y\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[\frac{y}{1+x^2}(\tan^{-1} x)^{y-1} - y^{\cot x}\cdot\operatorname{cosec}^2 x \log y\right]}{\left[(\tan^{-1} x)^y \log(\tan^{-1} x) + \cot x\, y^{\cot x - 1}\right]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[\frac{y}{1+x^2}(\tan^{-1} x)^{y-1} - y^{\cot x}\operatorname{cosec}^2 x \log y\right]}{\left[(\tan^{-1} x)^y \log(\tan^{-1} x) + \cot x\, y^{\cot x - 1}\right]}$$

**(1)**

## 4 marks Questions

**20.** If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$. **Delhi 2013C**

To prove, $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$

Given, $x = 2\cos\theta - \cos 2\theta$

and $y = 2\sin\theta - \sin 2\theta$

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$$

and $$\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta \quad \textbf{(1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2(\cos\theta - \cos 2\theta)}{2(-\sin\theta + \sin 2\theta)} \quad \textbf{(1)}$$

$$= \frac{2\sin\left(\frac{\theta+2\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)}{2\left[\cos\left(\frac{2\theta+\theta}{2}\right)\sin\left(\frac{2\theta-\theta}{2}\right)\right]} \quad \textbf{(1)}$$

$$\left[\begin{array}{l} \because \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right) \\ \text{and } \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right) \end{array}\right]$$

$$= \frac{\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)} = \tan\left(\frac{3\theta}{2}\right) = \text{RHS} \quad \textbf{(1)}$$

**Hence proved.**

**21.** If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$.

**Delhi 2013C, 2009; All India 2009C**

Given, $y = (\sin x)^x + \sin^{-1}\sqrt{x}$ ...(i)

Let $u = (\sin x)^x$ ...(ii)

Then, Eq. (i) becomes, $y = u + \sin^{-1}\sqrt{x}$ ...(iii)

On taking log both sides of Eq. (ii), we get

$$\log u = x \log \sin x \quad \textbf{(1)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log \sin x) + \log \sin x \frac{d}{dx}(x)$$

[by product rule]

$$\Rightarrow \frac{du}{dx} = u\left[x \times \frac{1}{\sin x}\frac{d}{dx}(\sin x) + \log \sin x (1)\right] \quad \textbf{(1)}$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x \left[\frac{x}{\sin x} \times \cos x + \log \sin x\right]$$

[from Eq. (ii)]

$$\Rightarrow \frac{du}{dx} = (\sin x)^x [x \cot x + \log \sin x] \quad \text{...(iv)}$$

On differentiating both sides of Eq.(iii) w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) \quad \textbf{(1)}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x [x \cot x + \log \sin x] + \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

[from Eq. (iv)] **(1)**

**22.** If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2.$$

**Delhi 2013C**

To prove, $x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

Given, $y = x \log\left(\frac{x}{a + bx}\right)$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = x\frac{d}{dx}\log\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\frac{d}{dx}(x)$$

[by product rule]

$$= x\left(\frac{1}{\frac{x}{a+bx}}\right)\frac{d}{dx}\left(\frac{x}{a+bx}\right) + \log\left(\frac{x}{a+bx}\right)\cdot 1$$

$$\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\right]$$

$$= (a+bx)\left[\frac{(a+bx)(1) - x(b)}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$

[by quotient rule]

$$= (a+bx)\left[\frac{a}{(a+bx)^2}\right] + \log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right) \quad \text{...(i)} \ \mathbf{(1)}$$

Now, $x\frac{dy}{dx} - y = \frac{ax}{a+bx} + x\log\left(\frac{x}{a+bx}\right) - y$

$$= \frac{ax}{a+bx} + x\log\left(\frac{x}{a+bx}\right) - x\log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx} \quad \text{...(ii)} \ \mathbf{(1)}$$

From Eq. (i), we have

$$\frac{dy}{dx} = \frac{a}{a+bx} + \log\left(\frac{x}{a+bx}\right)$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{-a}{(a+bx)^2} \cdot b + \frac{1}{\left(\frac{x}{a+bx}\right)} \cdot \left\{\frac{(a+bx)\cdot 1 - x \cdot b}{(a+bx)^2}\right\}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a+bx}{x}\left\{\frac{a+bx-bx}{(a+bx)^2}\right\}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a+bx}{x} \times \frac{a}{(a+bx)^2}$$

$$= \frac{-ab}{(a+bx)^2} + \frac{a}{x(a+bx)}$$ **(1)**

$$\Rightarrow x^3\frac{d^2y}{dx^2} = -\frac{abx^3}{(a+bx)^2} + \frac{ax^2}{(a+bx)}$$

[multiplying both sides by $x^3$]

$$= \frac{ax^2}{(a+bx)^2}\{-bx + (a+bx)\}$$

$$= \frac{a^2x^2}{(a+bx)^2}$$

$$\Rightarrow x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 = \left(x\frac{dy}{dx} - y\right)^2$$

[from Eq. (ii)] **(1)**

**Hence proved.**

**23.** Differentiate the following function with respect to $x$.

$$(\log x)^x + x^{\log x}$$ **Delhi 2013**

Let $y = (\log x)^x + x^{\log x}$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}\{(\log x)^x + x^{\log x}\}$$

$$= \frac{d}{dx}(\log x)^x + \frac{d}{dx}(x^{\log x})$$

$$= (\log x)^x \frac{d}{dx}[\{x \log(\log x)\}]$$

$$+ x^{\log x} \frac{d}{dx}(\log x \log x)$$

$$\left[\because \frac{d}{dx}(u^v) = u^v \frac{d}{dx}(v \log u)\right]$$ **(2)**

$$= (\log x)^x \left\{x\left(\frac{1}{\log x}\right)\frac{1}{x} + \log(\log x)\right\}$$

$$+ x^{\log x}\left[2(\log x)\frac{1}{x}\right]$$

$$\left[\because \frac{d}{dx}\log(\log x) = \frac{1}{\log x} \times \frac{1}{x},\right.$$

$$\left.\frac{d}{dx}(\log x \log x) = \frac{d}{dx}\{(\log x)^2\} = 2(\log x)\frac{1}{x}\right]$$

$$= (\log x)^x \left\{\frac{1}{\log x} + \log(\log x)\right\}$$

$$+ 2\left(\frac{\log x}{x}\right)x^{\log x}$$ **(2)**

**24.** If $y = \log[x + \sqrt{x^2 + a^2}]$, then show that

$$(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$ **Delhi 2013**

Given, $y = \log[x + \sqrt{x^2 + a^2}]$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \frac{d}{dx}\left(x + \sqrt{x^2 + a^2}\right)$$

$$\left[\because \frac{d}{dx}(\log x) = \frac{1}{x}\frac{d}{dx}(x)\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}}\left(1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right)$$

$$\left[\because \frac{d}{dx}(\sqrt{x^2 + a^2}) = \frac{1 \times 2x}{2\sqrt{x^2 + a^2}}\right]$$

$$= \frac{1}{x + \sqrt{x^2 + a^2}} \cdot \left(\frac{x + \sqrt{x^2 + a^2}}{\sqrt{x^2 + a^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}}\left(\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}(\sqrt{x^2 + a^2}) = 1 \quad \textbf{(1)}$$

Again on differentiating both sides w.r.t. to x, we get

$$\sqrt{x^2 + a^2}\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\sqrt{x^2 + a^2}) = \frac{d(1)}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}(\sqrt{x^2 + a^2}) + \frac{1 \cdot 2x \cdot \frac{dy}{dx}}{2\sqrt{x^2 + a^2}} = 0 \quad \textbf{(1)}$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}\right]$$

$$\Rightarrow \quad (x^2 + a^2) + \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0 \quad \textbf{(1)}$$

**Hence proved.**

**25.** Show that the function $f(x)=|x-3|, x\in R$, is continuous but not differentiable at $x=3$.

**Delhi 2013**

> Firstly, to check the differentiability of the function $f(x)$ at $x=3$. Find LHD and RHD, if LHD $\neq$ RHD, then function is not differentiable and then we check continuity of the function at $x=3$ by showing LHL = RHL = $f(3)$.

Given, $f(x)=|x-3|$

First, we check the differentiability of the given function $f(x)$ at $x=3$.

$$\text{LHD} = f'(3^-) = \lim_{h\to 0} \frac{f(3-h)-f(3)}{-h}$$

$$\left[\because Lf'(a) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}\right]$$

$$= \lim_{h\to 0} \frac{|3-h-3|-|3-3|}{-h}$$

$$= \lim_{h\to 0} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1 \qquad [\because |-x| = x] \ \textbf{(1)}$$

$$\text{RHD} = f'(3^+) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$\left[\because Rf'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right]$$

$$= \lim_{h\to 0} \frac{|3+h-3|-|3-3|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1 \qquad [\because |-x| = x] \ \textbf{(1)}$$

Since, LHD $\neq$ RHD at $x=3$

So, $f$ is not differentiable.

Now, we check the continuity of the given function $f(x)$ at $x=3$

$$\therefore \quad \text{LHL} = \lim_{x \to 3^-} |x-3|$$

$$= \lim_{h \to 0} |3-h-3| \qquad [\text{put } x = 3 - h]$$

$$= \lim_{h \to 0} |-h| = 0 \qquad \textbf{(1)}$$

$$\text{RHL} = \lim_{x \to 3^+} |x-3|$$

$$= \lim_{h \to 0} |3+h-3| = \lim_{h \to 0} |h| = 0 \qquad [\text{put } x = x + h]$$

and $f(3) = |3-3| = 0$

Thus, $\quad \text{LHL} = \text{RHL} = f(3)$

Hence, $f$ is continuous at $x=3$. **(1)**

**26.** If $x = a \sin t$ and $y = a(\cos t + \log \tan (t/2)$, then find $\frac{d^2y}{dx^2}$. **Delhi 2013**

Given, $y = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $x = a \sin t$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}\left(\log \tan \frac{t}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \frac{d}{dt}\left(\tan \frac{t}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dt} = a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}\right] \quad \textbf{(1)}$$

$$= a\left[-\sin t + \frac{1}{2} \times \frac{\cos\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)} \times \frac{1}{\cos^2\left(\frac{t}{2}\right)}\right]$$

$$= a\left[-\sin t + \frac{1}{2 \sin \frac{t}{2} . \cos \frac{t}{2}}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right] \qquad \left[\because \sin t = 2 \sin \frac{t}{2} . \cos \frac{t}{2}\right]$$

$$= a\left[\frac{1 - \sin^2 t}{\sin t}\right] = a \frac{\cos^2 t}{\sin t}$$

$[\because \sin^2 t + \cos^2 t = 1 \Rightarrow 1 - \sin^2 t = \cos^2 t]$ **(1)**

and $\frac{dx}{dt} = \frac{d}{dt}(a \sin t) = a \cos t$ **(1/2)**

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left[\frac{a \cos^2 t}{\sin t}\right]}{a \cos t} = \cot t$

$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}(\cot t)$ **(1/2)**

$$= \frac{d}{dt}(\cot t) \times \left(\frac{dt}{dx}\right) = (-\operatorname{cosec}^2 t)\left(\frac{dt}{dx}\right)$$

$$= -(\operatorname{cosec}^2 t).\frac{1}{a \cos t}$$

$$= -\frac{\operatorname{cosec}^2 t}{a \cos t}$$ **(1)**

**27.** Differentiate the following with respect to $x$

$$\sin^{-1}\left[\frac{2^{x+1} \cdot 3^x}{1+(36)^x}\right].$$

**HOTS; All India 2013**

Firstly, put $6^x$ equal to $\tan\theta$, so that it becomes to some standard trigonometric function. Then, simplify the expression and then differentiate by chain rule.

Let $\quad y = \sin^{-1}\left[\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

$$= \sin^{-1}\left(\frac{2\cdot 2^x\cdot 3^x}{1+(6^2)^x}\right)$$

$$= \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right] \qquad \textbf{(1)}$$

Put $\tan\theta = 6^x \Rightarrow \theta = \tan^{-1}(6^x)$

Then, $\quad y = \sin^{-1}\left(\dfrac{2\cdot\tan\theta}{1+\tan^2\theta}\right)$

$$= \sin^{-1}(\sin 2\theta)\left[\because \sin\ 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right] \qquad \textbf{(1)}$$

$$= 2\theta$$

$\Rightarrow y = 2\tan^{-1}(6^x)$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{2}{1+(6^x)^2}\frac{d}{dx}(6^x)\left[\because \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}\right]$$

**(1)**

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+(6^x)^2}\cdot 6^x\cdot\log 6$$

$$= \left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]\log 6 \qquad \textbf{(1)}$$

**28.** If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{6}$. **All India 2013**

Given, $x = a\cos^3\theta$ and $y = a\sin^3\theta$

On differentiating both sides w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = 3a\cos^2\theta \frac{d}{d\theta}(\cos\theta)$$

$$= 3a\cos^2\theta \cdot (-\sin\theta)$$

$$= -3a\cos^2\theta \cdot \sin\theta \qquad \textbf{(1)}$$

and $$\frac{dy}{d\theta} = 3a\sin^2\theta \frac{d}{d\theta}(\sin\theta)$$

$$= 3a\sin^2\theta \cdot (\cos\theta) = 3a\sin^2\theta \cdot \cos\theta$$

Now, $$\frac{dy}{dx} = \left(\frac{dy/d\theta}{dx/d\theta}\right)$$

$$= \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta} = -\tan\theta \qquad \textbf{(1)}$$

Again, on differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = -\frac{d}{d\theta}(\tan\theta)\frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \frac{d\theta}{dx}$$

$$= -\sec^2\theta \cdot \left(\frac{-1}{3a\cos^2\theta \cdot \sin\theta}\right)$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{1}{3a\cos^4\theta \cdot \sin\theta} \qquad \textbf{(1)}$$

At

$$\theta = \frac{\pi}{6}, \left(\frac{d^2y}{dx^2}\right)_{\left(\theta = \frac{\pi}{6}\right)} = \frac{1}{3a\left(\cos\frac{\pi}{6}\right)^4\left(\sin\frac{\pi}{6}\right)}$$

$$= \frac{1}{3a\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}$$

$$= \frac{1}{3a\left(\frac{9}{16}\right)\left(\frac{1}{2}\right)} = \frac{32}{27a} \qquad \textbf{(1)}$$

**29.** If $x\sin(a+y)+\sin a\cos(a+y)=0$, then prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$. **All India 2013**

To prove, $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$

Given, $x\sin(a+y)+\sin a\cos(a+y)=0$

$$\Rightarrow \quad x=\frac{-\sin a\cos(a+y)}{\sin(a+y)} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. $y$, we get

$$\frac{dx}{dy}=\frac{-\left[\begin{array}{l}\sin(a+y)\frac{d}{dy}\{\sin a\cos(a+y)\}\\ -\sin a\cos(a+y)\frac{d}{dy}\{\sin(a+y)\}\end{array}\right]}{\sin^2(a+y)}$$

[using quotient rule]

$$=\left\{\frac{\begin{array}{l}\sin(a+y)\cdot\sin a\sin(a+y)\\ \qquad +\sin a\cos(a+y)\cos(a+y)\end{array}}{\sin^2(a+y)}\right\} \textbf{(1)}$$

$$=\frac{\sin a}{\sin^2(a+y)}\{\sin^2(a+y)+\cos^2(a+y)\}$$

$$=\frac{\sin a}{\sin^2(a+y)}\cdot 1 \quad [\because \sin^2\theta+\cos^2\theta=1] \; \textbf{(1)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$ **Hence proved. (1)**

**30.** If $x^y=e^{x-y}$, then prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$ or $\frac{dy}{dx}=\frac{\log x}{\{\log(xe)\}^2}$.

**All India 2013, 2011, 2010**

Firstly, take log on both sides and convert it into $y = f(x)$ form. Then, differentiate both sides by quotient rule to get required result.

To prove, $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$

Given $x^y = e^{x-y}$

On taking log both sides, we get

$$y \log_e x = (x-y) \log_e e \quad \textbf{(1)}$$

$$\Rightarrow \quad y \log_e x = x - y \quad [\because \log_e e = 1]$$

$$\Rightarrow \quad y(1+\log x) = x$$

$$\Rightarrow \quad y = \frac{x}{1+\log x} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{v}{u}\right) = \frac{u\frac{dv}{dx} - v\frac{du}{dx}}{u^2}\right] \quad \textbf{(1)}$$

$$= \frac{1+\log x - x \cdot \frac{1}{x}}{(1+\log x)^2}$$

$$= \frac{1 + \log x - 1}{(1 + \log x)^2}$$

$$= \frac{\log x}{(1 + \log x)^2}$$ **Hence proved. (1)**

Also, it can be written as

$$\frac{dy}{dx} = \frac{\log x}{(\log_e e + \log x)^2} \quad [\because \log_e e = 1]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{\{\log(ex)\}^2}$$

**31.** If $y^x = e^{y-x}$, then prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$.

**All India 2013**

To prove, $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$

Given that $y^x = e^{y-x}$

On taking log both sides, we get

$$\log y^x = \log e^{(y-x)}$$

$$\Rightarrow \quad x\log y = (y-x)\log e$$

$$\Rightarrow \quad x\log y = y-x \qquad [\because \log e = 1] \quad \text{...(i)} \ \mathbf{(1)}$$

On differentiating both sides w.r.t. $x$, we get

$$x\cdot\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)=\frac{d}{dx}(y)-\frac{d}{dx}(x)$$

$$\Rightarrow \quad x\cdot\frac{1}{y}\cdot\frac{dy}{dx}+\log y\cdot 1=\frac{dy}{dx}-1$$

$$\Rightarrow \quad (1+\log y)=\frac{dy}{dx}\left(1-\frac{x}{y}\right)$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y(1+\log y)}{(y-x)} \qquad \text{...(ii)} \ \mathbf{(1)}$$

On putting the value of $x$ from Eq. (i) in Eq. (ii), we get

$$\frac{dy}{dx}=\frac{y(1+\log y)}{y-\left(\frac{y}{1+\log y}\right)}=\frac{y(1+\log y)^2}{(y+y\log y-y)} \qquad \mathbf{(1)}$$

$$=\frac{y(1+\log y)^2}{y\log y}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{(1+\log y)^2}{\log y} \qquad \mathbf{(1)}$$

**Hence proved.**

**32.** If $(\cos x)^y = (\cos y)^x$, then find $\frac{dy}{dx}$.

**HOTS; Delhi 2012**

Firstly, take log on both sides, then differentiate both sides by product rule.

Given, $(\cos x)^y = (\cos y)^x$

On taking log both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\Rightarrow \quad y\log(\cos x) = x\log(\cos y)$$

$[\because \log x^n = n\log x]$ **(1)**

On differentiating both sides w.r.t. $x$, we get

$$y\cdot\frac{d}{dx}\log(\cos x) + \log\cos x\cdot\frac{d}{dx}(y)$$

$$= x\frac{d}{dx}\log(\cos y) + \log(\cos y)\frac{d}{dx}(x)$$

$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$ **(1)**

$$\Rightarrow \quad y\cdot\frac{1}{\cos x}\frac{d}{dx}(\cos x) + \log(\cos x)\frac{dy}{dx}$$

$$= x\cdot\frac{1}{\cos y}\frac{d}{dx}(\cos y) + \log\cos y\cdot 1$$

$$\Rightarrow \quad y.\frac{1}{\cos x}(-\sin x) + \log(\cos x)\cdot\frac{dy}{dx}$$

$$= x\frac{1}{\cos y}(-\sin y)\cdot\frac{dy}{dx} + \log(\cos y)\cdot 1$$ **(1)**

$$\Rightarrow -y\tan x + \log(\cos x)\frac{dy}{dx} = -x\tan y\frac{dy}{dx} + \log(\cos y)$$

$$\Rightarrow [x\tan y + \log(\cos x)]\frac{dy}{dx} = \log(\cos y) + y\tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log(\cos y) + y\tan x}{x\tan y + \log(\cos x)}$$ **(1)**

**33.** If $\sin y = x\sin(a+y)$, then prove that

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}.$$

**HOTS; Delhi 2012**

In the given expression, we collect all the terms of $y$ on RHS and a term $x$ on LHS and then differentiate with respect to $y$ on both sides to get required result.

Given, $\sin y = x \sin(a + y)$

$$\Rightarrow \quad x = \frac{\sin y}{\sin(a + y)}$$

On differentiating both sides w.r.t. $y$, we get

$$\frac{dx}{dy} = \frac{\sin(a + y).\frac{d}{dy}(\sin y) - \sin y.\frac{d}{dy}\sin(a + y)}{\sin^2(a + y)}$$

[by using quotient rule] **(1½)**

$$= \frac{\sin(a + y)\cos y - \sin y \cos(a + y)}{\sin^2(a + y)}$$

$$= \frac{\sin(a + y - y)}{\sin^2(a + y)}$$ **(1½)**

[$\because \sin A \cos B - \cos A \sin B = \sin(A - B)$]

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin a}{\sin^2(a + y)}$$

Hence, $\dfrac{dy}{dx} = \dfrac{\sin^2(a + y)}{\sin a}$ **(1)**

**Hence proved.**

**NOTE** As the result is in $y$ form, so we consider here $x$ as a dependent variable.

**34.** If $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$, then show that $\dfrac{dy}{dx} = \dfrac{-y}{x}$.

**All India 2012**

Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

To show, $\dfrac{dy}{dx} = \dfrac{-y}{x}$

Now $x = (a^{\sin^{-1} t})^{1/2}$

Now, $x = (a^{\sin^{-1} t})^{1/2}$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dx}{dt} = \frac{1}{2}(a^{\sin^{-1} t})^{-1/2} \frac{d}{dt}(a^{\sin^{-1} t})$$

$$\left[\because \frac{d}{dx}(x^n) = nx^{n-1}\right]$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{-1/2}\, a^{\sin^{-1} t} \log a \frac{d}{dt}(\sin^{-1} t)$$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{-1/2}\, a^{\sin^{-1} t} \log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$= \frac{1}{2}(a^{\sin^{-1} t})^{1/2} \log a \cdot \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\frac{1}{2}\sqrt{a^{\sin^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \qquad \text{...(i)} \; \textbf{(1½)}$$

And $y = (a^{\cos^{-1} t})^{1/2}$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = \frac{1}{2}(a^{\cos^{-1} t})^{-1/2} \frac{d}{dt}(a^{\cos^{-1} t})$$

$$\left[\because \frac{d}{dx} x^n = nx^{n-1}\right]$$

$$= \frac{1}{2}(a^{\cos^{-1} t})^{-1/2} a^{\cos^{-1} t} \log a \frac{d}{dt}(\cos^{-1} t)$$

$$= \frac{1}{2}(a^{\cos^{-1} t})^{1/2} \log a \cdot \frac{(-1)}{\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-\frac{1}{2}\sqrt{a^{\cos^{-1} t}} \cdot \log a}{\sqrt{1-t^2}} \qquad \text{...(ii)} \quad \textbf{(1½)}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{\left(\frac{-\frac{1}{2}\sqrt{a^{\cos^{-1}t}}\log a}{\sqrt{1-t^2}}\right)}{\left(\frac{\frac{1}{2}\sqrt{a^{\sin^{-1}t}}\log a}{\sqrt{1-t^2}}\right)}$$

$$=-\frac{\sqrt{a^{\cos^{-1}t}}}{\sqrt{a^{\sin^{-1}t}}}=-\frac{y}{x} \quad \textbf{(1)}$$

$\left[\because \text{ given } \sqrt{a^{\cos^{-1}t}}=y \text{ and } \sqrt{a^{\sin^{-1}t}}=x\right]$

**Hence proved.**

**35.** Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ w.r.t. $x$.

HOTS; All India 2012

Firstly, put $x = \tan\theta$ and convert $y$ in terms of $\theta$, then put $\theta = \tan^{-1} x$ and differentiate w.r.t. x.

Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$, we get

$$y = \tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta} \quad \textbf{(1)}$$

$$y = \tan^{-1}\left[\frac{\sec\theta - 1}{\tan\theta}\right] \quad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \tan^{-1}\left[\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right]$$

$$\left[\because \sec\theta = \frac{1}{\cos\theta} \text{ and } \tan\theta = \frac{\sin\theta}{\cos\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos\theta}{\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right]$$

$$\left[\because 1-\cos x = 2\sin^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right]$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} \quad [\because \tan^{-1}(\tan\phi) = \phi]$$

$$\Rightarrow \quad y = \frac{\tan^{-1} x}{2} \quad [\because \theta = \tan^{-1} x] \ \textbf{(1½)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{1+x^2} \left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

Hence, $\frac{dy}{dx} = \frac{1}{2(1+x^2)}$ **(1½)**

**36.** If $y = (\tan^{-1} x)^2$, then show that

$$(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$

Delhi 2012

Given, $y = (\tan^{-1} x)^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x . \frac{1}{1+x^2} \quad \left[\because \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\tan^{-1} x}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = 2\tan^{-1} x$$ **(1½)**

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\cdot\frac{d}{dx}(1+x^2) = \frac{d}{dx}(2\tan^{-1} x)$$ **(1)**

$$\Rightarrow \quad (1+x^2).\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 2x = \frac{2}{1+x^2}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}(1+x^2) = 2$$

$$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$$ **(1½)**

**Hence proved.**

**37.** If $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$, then find $\frac{dy}{dx}$.

Delhi 2012C

Given, $y = x^{\sin x - \cos x} + \dfrac{x^2 - 1}{x^2 + 1}$

Let $u = x^{\sin x - \cos x}$

and $v = \dfrac{x^2 - 1}{x^2 + 1}$

Consider $u = x^{\sin x - \cos x}$

On taking log both sides, we get

$$\log u = (\sin x - \cos x) \cdot \log x$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\frac{du}{dx} = (\sin x - \cos x) \cdot \frac{1}{x} + \log x \cdot (\cos x + \sin x)$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x - \cos x} \left[\frac{\sin x - \cos x}{x} + \log x \cdot (\cos x + \sin x)\right]$$

**(1½)**

Now, consider $v = \dfrac{x^2 - 1}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dv}{dx} = 0 - \frac{(x^2 + 1)\frac{d}{dx}(2) - 2\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$\Rightarrow \frac{dv}{dx} = -\left[\frac{0 - 2 \cdot 2x}{(x^2 + 1)^2}\right] = \frac{4x}{(x^2 + 1)^2} \quad \textbf{(1½)}$$

Now, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

$$= x^{\sin x - \cos x}\left[\frac{\sin x - \cos x}{x} + \log x(\cos x + \sin x) + \frac{4x}{(x^2 + 1)^2}\right] \quad \textbf{(1)}$$

**38.** If $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$, then find $\frac{d^2y}{dx^2}$ and $\frac{d^2y}{dt^2}$. **Delhi 2012C**

Given, $x=a(\cos t+t\sin t)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dx}{dt}=a(-\sin t+1\cdot\sin t+t\cos t) \quad \textbf{(1)}$$

$$=at\cos t$$

Also given, $y=a(\sin t-t\cos t)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt}=a(\cos t-\cos t\cdot 1+t\sin t)=at\sin t \quad \text{...(i)} \quad \textbf{(1)}$$

Now, $$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{at\sin t}{at\cos t}=\tan t$$

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}(\tan t)\,\frac{dt}{dx}=\sec^2 t\,\frac{1}{dx/dt}$$

$$=\frac{\sec^2 t}{at\cos t}=\frac{\sec^3 t}{at} \quad \textbf{(1)}$$

Also,

$$\frac{d^2y}{dt^2}=\frac{d}{dt}(at\sin t)$$

$$=a(\sin t+t\cos t) \quad \textbf{(1)}$$

**39.** Find $\frac{dy}{dx}$, when $y=x^{\cot x}+\frac{2x^2-3}{x^2+x+2}$.

**All India 2012C**

Given, $y=x^{\cot x}+\frac{2x^2-3}{x^2+x+2}$

Let $u=x^{\cot x}$ and $v=\frac{2x^2-3}{x^2+x+2}$

Consider $u=x^{\cot x}$

Consider $u = x^{\cot x}$

On taking log both sides, we get

$$\log u = \cot x \log x$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \cot x \cdot \frac{1}{x} - \operatorname{cosec}^2 x \cdot \log x$$

$$\Rightarrow \quad \frac{du}{dx} = u\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right)$$

$$= x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) \quad \textbf{(1½)}$$

Now, consider $v = \dfrac{2x^2 - 3}{x^2 + x + 2}$

On differentiating both sides w.r.t. x, we get

$$\frac{dv}{dx} = \frac{(x^2+x+2)(4x) - (2x^2-3)(2x+1)}{(x^2+x+2)^2}$$

$$= \frac{4x^3 + 4x^2 + 8x - 4x^3 - 2x^2 + 6x + 3}{(x^2+x+2)^2}$$

$$= \frac{2x^2 + 14x + 3}{(x^2+x+2)^2} \quad \textbf{(1½)}$$

Now, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$

$$= x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) + \frac{2x^2 + 14x + 3}{(x^2+x+2)^2} \quad \textbf{(1)}$$

**40.** If $x = \cos t + \log \tan \frac{t}{2}$ and $y = \sin t$, then find the values of $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{4}$. **All India 2012C**

Given, $x = \cos t + \log \tan\left(\frac{t}{2}\right)$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dx}{dt} = -\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)} \cdot \sec^2 \frac{t}{2} \cdot \frac{1}{2}$$

$$= -\sin t + \frac{\cos(t/2)}{\sin(t/2)} \cdot \frac{1}{\cos^2\left(\frac{t}{2}\right)} \cdot \frac{1}{2}$$

$$= -\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}$$

$$= -\sin t + \frac{1}{\sin t} = \frac{-\sin^2 t + 1}{\sin t} = \frac{\cos^2 t}{\sin t} \quad \textbf{(1)}$$

$$\left[\because 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\right]$$

Also given, $y = \sin t$

On differentiating both sides w.r.t. $t$, we get

$$\frac{dy}{dt} = \cos t$$

Again, differentiating both sides w.r.t. $t$, we get

$$\frac{d^2y}{dt^2} = -\sin t \quad \textbf{(1)}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{\frac{\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \sec^2 t \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{\sin t}{\cos^2 t} = \sec^4 t \sin t$$

At $t = \frac{\pi}{4}$, $\frac{d^2y}{dt^2} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$ **(1)**

At $t = \frac{\pi}{4}$, $\frac{d^2y}{dx^2} = \sec^4\frac{\pi}{4} \cdot \sin\frac{\pi}{4}$

$$= (\sqrt{2})^4 \cdot \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \quad \textbf{(1)}$$

**41.** If $x\sqrt{1+y} + y\sqrt{1+x} = 0, (x \neq y)$, then prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$.

**HOTS; Foreign 2012; Delhi 2011C**

Firstly, solve the given equation and convert it into $y = f(x)$ form. Then, differentiate to get the required result.

To prove, $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Given equation is $x\sqrt{1+y} + y\sqrt{1+x} = 0$, where $x \neq y$, we first convert the given equation into $y = f(x)$ form.

So, $$x\sqrt{1+y} = -y\sqrt{1+x}$$

On squaring both sides, we get

$$x^2(1+y) = y^2(1+x)$$

$$\Rightarrow \quad x^2 + x^2y = y^2 + y^2x$$

$$\Rightarrow \quad x^2 - y^2 = y^2x - x^2y$$

$\Rightarrow \quad (x-y)(x+y) = -xy(x-y)$

$[\because a^2 - b^2 = (a-b)(a+b)]$

$\Rightarrow \quad (x-y)(x+y) + xy(x-y) = 0$

$\Rightarrow \quad (x-y)(x+y+xy) = 0$

$\therefore$ Either $x - y = 0$ or $x + y + xy = 0$

Now, $\quad x - y = 0 \Rightarrow x = y$

But it is given that $x \neq y$.

So, we get a contradiction.

$\Rightarrow \quad x - y = 0$ is rejected. **(1)**

$\therefore \quad y + xy + x = 0 \Rightarrow y(1+x) = -x$

$\Rightarrow \quad y = \frac{-x}{1+x}$ ...(i) **(1)**

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{(1+x) \times \frac{d}{dx}(-x) - (-x) \times \frac{d}{dx}(1+x)}{(1+x)^2}$$ **(1)**

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+x)(-1) + x(1)}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1 - x + x}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{(1+x)^2}$$ **(1)**

**Hence proved**

**42.** If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that

$(1+x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0.$ **All India 2011**

Given, $x = \tan\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \quad \tan^{-1} x = \frac{1}{a}\log y$$

$[\because \tan\theta = a \Rightarrow \theta = \tan^{-1} a]$

$$\Rightarrow \quad a\tan^{-1} x = \log y$$

On differentiating both sides w.r.t. $x$, we get

$$a \times \frac{1}{1+x^2} = \frac{1}{y}\cdot\frac{dy}{dx} \qquad \textbf{(1)}$$

$$\left[\because \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay \qquad \textbf{(1)}$$

Again, on differentiating both sides w.r.t. $x$, we get

$$(1+x^2)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dx}(1+x^2) = \frac{d}{dx}(ay)$$

$$\left[\because \frac{d}{dx}(u\cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right] \textbf{(1)}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot(2x) = a\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0 \qquad \textbf{(1)}$$

**43.** Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. $x$.

**Delhi 2011**

Let $u = x^{x \cos x}$ and $v = \frac{x^2+1}{x^2-1}$, then $y = u + v$.

Now, find $\frac{du}{dx}$ and $\frac{dv}{dx}$. Then, put these values in $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Let $\quad y = x^{x \cos x} + \frac{x^2+1}{x^2-1}$

Again let $\quad u = x^{x \cos x}$ and $v = \frac{x^2+1}{x^2-1}$

Then, $\quad y = u + v$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots\text{(i)} \ \mathbf{(1)}$$

Now, $\quad u = x^{x \cos x}$

On taking log both sides, we get

$$\log u = \log x^{x \cos x}$$

$\Rightarrow \log u = (x \cos x) \cdot \log x \quad [\because \log m^n = n \log m]$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\frac{du}{dx} = (x \cos x) \times \frac{d}{dx}(\log x) + \log x \times \frac{d}{dx}(x \cos x)$$

[by product rule]

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cos x \times \frac{1}{x} + \log x \cdot [-x \sin x + \cos x]$$

$$\left[\begin{aligned} \because \frac{d}{dx}(x \cos x) &= x \times \frac{d}{dx}\cos x + \cos x \frac{d}{dx} x \\ &= x(-\sin x) + \cos x \cdot 1 \\ &= -x \sin x + \cos x \end{aligned}\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \cos x - x \log x \sin x + \log x \cos x$$

$$\Rightarrow \frac{du}{dx} = u\ [\cos x - x \log x \sin x + \log x \cos x]$$

$$\Rightarrow \frac{du}{dx} = x^{x \cos x}[\cos x - x \log x \sin x + \log x \cos x]\ [\because u = x^{x \cos x}]$$ **(1½)**

and $v = \frac{x^2+1}{x^2-1}$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{(x^2-1) \times \frac{d}{dx}(x^2+1) - (x^2+1) \times \frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

[by quotient rule]

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2-1) \cdot 2x - (x^2+1) \cdot 2x}{(x^2-1)^2}$$

$$= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{-4x}{(x^2-1)^2}$$

On putting values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in Eq. (i), we get

$$\frac{dy}{dx} = x^{x \cos x}[\cos x - x \log x \sin x + \log x \cos x] - \frac{4x}{(x^2-1)^2}$$ **(1½)**

**44.** If $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$, then find $\frac{d^2y}{dx^2}$.

**Delhi 2011**

Here, we use chain rule, i.e. if $y = f_1(\theta)$ and $x = f_2(\theta)$, then $\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$ to get required value.

Given, $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$

On differentiating both sides w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \left[\because \frac{d}{d\theta}\sin\theta = \cos\theta\right]$$

and $$\frac{dy}{d\theta} = -a\sin\theta \left[\because \frac{d}{d\theta}\cos\theta = -\sin\theta\right] \quad \textbf{(1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)}$$

$$= \frac{-a\sin\theta}{a(1 - \cos\theta)} = \frac{-\sin\theta}{1 - \cos\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} = -\cot\frac{\theta}{2} \quad \textbf{(1)}$$

$$\left[\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \text{ and } 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

Again differentiating w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\cot\frac{\theta}{2}\right)$$

$$= \frac{d}{d\theta}\left(-\cot\frac{\theta}{2}\right) \times \frac{d\theta}{dx}$$

$$\left[\because \frac{d}{dx}[f(\theta)] = \frac{d}{d\theta} f(\theta) \times \frac{d\theta}{dx}\right]$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{a(1-\cos\theta)} \quad \textbf{(1)}$$

$$\left[\because \frac{d}{d\theta}\cot\theta = -\operatorname{cosec}^2\theta\right]$$

$$= \frac{1}{2a}\operatorname{cosec}^2\frac{\theta}{2} \times \frac{1}{2\sin^2\frac{\theta}{2}}$$

$$\left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2} \quad \textbf{(1)}$$

**45.** Prove that

$$\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = \sqrt{a^2 - x^2}.$$

**Foreign 2011**

To prove $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$

$$=\sqrt{a^2-x^2}$$

$$\text{LHS}=\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$=\left[\frac{x}{2}\times\frac{d}{dx}\sqrt{a^2-x^2}+\sqrt{a^2-x^2}\times\frac{d}{dx}\left(\frac{x}{2}\right)+\frac{a^2}{2}\times\frac{d}{dx}\sin^{-1}\frac{x}{a}\right] \quad \textbf{(1)}$$

$$\left[\because \frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx}\right]$$

$$=\frac{x}{2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\frac{d}{dx}(a^2-x^2)+\sqrt{a^2-x^2}\cdot\frac{1}{2}+\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{d}{dx}\left(\frac{x}{a}\right)$$

$$=\left[\frac{x}{2}\cdot\frac{-2x}{2\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}\cdot\frac{1}{2}+\frac{a^2}{2}\times\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\right] \quad \textbf{(1)}$$

$$=\frac{-x^2}{2\sqrt{a^2-x^2}}+\frac{\sqrt{a^2-x^2}}{2}+\frac{a^2}{2a}\cdot\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}}$$

$$=\frac{-x^2}{\ldots}+\frac{\sqrt{a^2-x^2}}{\ldots}+\frac{a^2}{\ldots}\times\frac{a}{\ldots}$$

$$= \frac{-x^2}{2\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2\sqrt{a^2-x^2}} \quad \textbf{(1)}$$

$$= \frac{-x^2 + (a^2 - x^2) + a^2}{2\sqrt{a^2-x^2}}$$

$$= \frac{2a^2 - 2x^2}{2\sqrt{a^2-x^2}} = \frac{2(a^2-x^2)}{2\sqrt{a^2-x^2}}$$

$$= \frac{a^2-x^2}{(a^2-x^2)^{1/2}} = (a^2-x^2)^{1/2} = \sqrt{a^2-x^2}$$

= RHS **Hence proved. (1)**

**46.** If $y = \log[x + \sqrt{x^2+1}]$, then prove that

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

**Foreign 2011**

Do same as Que 24.
[**Hint** put $a = 1$ in Que. 24]

**47.** If $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$, then show that $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$.

**All India 2011C**

To prove, $(1+x^2)\frac{dy}{dx} + xy + 1 = 0$

Given, $\log(\sqrt{1+x^2} - x) = y\sqrt{1+x^2}$ ...(i)

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{\sqrt{1+x^2}-x}\frac{d}{dx}[\sqrt{1+x^2}-x]$$

$$= y\frac{d}{dx}\sqrt{1+x^2} + \sqrt{1+x^2}\frac{dy}{dx} \quad \textbf{(1)}$$

[by chain rule]

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{1}{2\sqrt{1+x^2}}\frac{d}{dx}(x^2)-1\right]$$

$$= \frac{y}{2\sqrt{1+x^2}}\frac{d}{dx}(1+x^2)+\sqrt{1+x^2}\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{2x}{2\sqrt{1+x^2}}-1\right]$$

$$= y \times \frac{2x}{2\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot\frac{dy}{dx} \quad \textbf{(1½)}$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}-x}\left[\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\right]$$

$$= \frac{xy}{\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot\frac{dy}{dx}$$

$$\Rightarrow \frac{-1}{\sqrt{1+x^2}} = \frac{xy+(1+x^2)\frac{dy}{dx}}{\sqrt{1+x^2}}$$

$$\Rightarrow -1 = xy+(1+x^2)\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}+xy+1=0 \quad \textbf{(1½)}$$

**Hence proved.**

**48.** If $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$, then find $\frac{d^2y}{dx^2}$.

All India 2011C

Given,

$x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$ ...(i)

On differentiating both sides w.r.t. x, we get

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad \left[\because \frac{d}{d\theta}\sin\theta = \cos\theta\right]$$

and $$\frac{dy}{d\theta} = a\sin\theta \quad \left[\because \frac{d}{d\theta}\cos\theta = -\sin\theta\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$\left[\begin{array}{l}\because \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ \text{and } 1 + \cos x = 2\cos^2\frac{x}{2}\end{array}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \tan\frac{\theta}{2}$$ **(1)**

Again, differentiating w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{\theta}{2}\right) \quad \textbf{(1/2)}$$

$$\left[\because \frac{d}{dx} f(\theta) = \frac{d}{d\theta} f(\theta) \cdot \frac{d\theta}{dx}\right]$$

$$= \frac{d}{d\theta}\left(\tan\frac{\theta}{2}\right) \times \frac{d\theta}{dx} = \sec^2\frac{\theta}{2} \cdot \frac{d}{d\theta}\left(\frac{\theta}{2}\right) \cdot \frac{d\theta}{dx}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{1}{a(1+\cos\theta)} \quad \textbf{(1)}$$

$$= \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{1}{a \times 2\cos^2\frac{\theta}{2}}$$

$$\left[1 + \cos x = 2\cos^2\frac{x}{2}\right]$$

$$= \frac{1}{4a}\sec^4\frac{\theta}{2} \quad \textbf{(1½)}$$

**49.** If $y = a\sin x + b\cos x$, then prove that

$$y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2.$$

**All India 2011C; HOTS**

Firstly, we differentiate the given expression with respect to x and get first derivative of y. Now, put the value of y and first derivative of y in LHS of given expression and then solve it and get required RHS.

To prove, $y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2$ ...(i)

Given, $y = a \sin x + b \cos x$ ...(ii)

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\frac{dy}{dx} = a \cos x - b \sin x \quad \textbf{(1)}$$

Now, we take LHS of Eq. (i), we get

$$\text{LHS} = y^2 + \left(\frac{dy}{dx}\right)^2$$

On putting the value of y and $dy/dx$, we get

$$\text{LHS} = (a \sin x + b \cos x)^2 + (a \cos x - b \sin x)^2$$
$$= a^2 \sin^2 x + b^2 \cos^2 x + 2ab \sin x \cos x + a^2 \cos^2 x + b^2 \sin^2 x - 2ab \sin x \cos x \quad \textbf{(1½)}$$
$$[\because (a \pm b)^2 = a^2 + b^2 \pm 2ab]$$
$$= a^2 \sin^2 x + b^2 \cos^2 x + a^2 \cos^2 x + b^2 \sin^2 x$$
$$= a^2 (\sin^2 x + \cos^2 x) + b^2 (\sin^2 x + \cos^2 x)$$
$$= a^2 + b^2 \quad [\because \sin^2 x + \cos^2 x = 1]$$
$$= \text{RHS}$$

**Hence proved. (1½)**

**50.** If $x = a(\cos\theta + \theta \sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, then find $\frac{d^2y}{dx^2}$.

All India 2011C, 2008

Do same as Que 38 $\left[\textbf{Ans. } 7\frac{\sec^3\theta}{a\theta}\right]$

**51.** If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

**Delhi 2011C**

Given, $x = a(\theta - \sin\theta)$

and $y = a(1 + \cos\theta)$

On differentiating both sides w.r.t. $\theta$, we get

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = -a\sin\theta \quad \textbf{(1)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1 - \cos\theta)} \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{a \times 2\sin^2\frac{\theta}{2}}$$

$$\left[\because \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \text{ and } 1 - \cos x = 2\sin^2\frac{x}{2}\right] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$

On putting $\theta = \frac{\pi}{3}$, we get

$$\left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{3}} = -\cot\frac{\pi}{6} = -\sqrt{3}$$

$$\left[\because \cot\frac{\pi}{6} = \sqrt{3}\right] \quad \textbf{(1)}$$

Hence, $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$ is $-\sqrt{3}$.

**52.** If $y = (\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$, then find $\frac{dy}{dx}$.

**All India 2010C**

Firstly, take log on both sides and then differentiate to get the value of $\frac{dy}{dx}$.

Given, $y = (\sin x - \cos x)^{(\sin x - \cos x)}, \frac{\pi}{4} < x < \frac{3\pi}{4}$

On taking log both sides, we get

$$\log y = \log (\sin x - \cos x)^{(\sin x - \cos x)}$$

$$\Rightarrow \log y = (\sin x - \cos x) \cdot \log (\sin x - \cos x)$$

$$[\because \log m^n = n \log m] \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = (\sin x - \cos x) \times \frac{d}{dx} \log (\sin x - \cos x)$$

$$+ \log (\sin x - \cos x) \times \frac{d}{dx} (\sin x - \cos x)$$

**[by product rule]**

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x) \cdot \frac{1}{(\sin x - \cos x)} \cdot \frac{d}{dx}(\sin x - \cos x) + \log(\sin x - \cos x) \cdot (\cos x + \sin x) \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\sin x - \cos x)\frac{1}{(\sin x - \cos x)} (\cos x + \sin x) + \log(\sin x - \cos x) \cdot (\cos x + \sin x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = (\cos x + \sin x) + (\cos x + \sin x) \cdot [\log(\sin x - \cos x)]$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{dy}{dx} = (\cos x + \sin x) [1 + \log(\sin x - \cos x)] \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y(\cos x + \sin x) [1 + \log(\sin x - \cos x)]$$

$$\therefore \quad \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)} \cdot (\cos x + \sin x) [1 + \log(\sin x - \cos x)]$$

**(1)**

**53.** If $y = \cos^{-1}\left[\dfrac{2x - 3\sqrt{1 - x^2}}{\sqrt{13}}\right]$, then find $\dfrac{dy}{dx}$.

**All India 2010C**

In the given expression, put $x = \sin\theta$ and simplify the resulting expression, then differentiate it.

Given, $y = \cos^{-1}\left[\dfrac{2x - 3\sqrt{1-x^2}}{\sqrt{13}}\right]$

Put $\quad x = \sin\theta$, then $\theta = \sin^{-1} x$

$$\left[\begin{array}{l}\because \text{ for } \sqrt{a^2 - x^2}, \text{ we put } x = a\sin\theta \\ \therefore \text{ for } \sqrt{1 - x^2}, \text{ we put } x = \sin\theta\end{array}\right] \quad \textbf{(1)}$$

$$\therefore \quad y = \cos^{-1}\left[\frac{2\sin\theta - 3\sqrt{1-\sin^2\theta}}{\sqrt{13}}\right]$$

$$\Rightarrow \quad y = \cos^{-1}\left[\frac{2\sin\theta - 3\cos\theta}{\sqrt{13}}\right]$$

$$[\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$$

$$\Rightarrow y = \cos^{-1}\left[\frac{2}{\sqrt{13}}\sin\theta - \frac{3}{\sqrt{13}}\cos\theta\right]$$

Now, let $\frac{2}{\sqrt{13}} = \cos\alpha$ and $\frac{3}{\sqrt{13}} = \sin\alpha$ **(1½)**

$$\left[\because \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{\sqrt{13}}\right)^2 + \left(\frac{2}{\sqrt{13}}\right)^2 = \frac{9}{13} + \frac{4}{13} = \frac{13}{13} = 1\right]$$

$\therefore \quad y = \cos^{-1}[\sin\theta\cos\alpha - \cos\theta\sin\alpha]$

$\Rightarrow \quad y = \cos^{-1}\sin(\theta - \alpha)$

$[\because \sin\theta\cos\alpha - \cos\theta\sin\alpha = \sin(\theta - \alpha)]$

$$\Rightarrow \quad y = \cos^{-1}\cos\left[\frac{\pi}{2} - (\theta - \alpha)\right]$$

$$\left[\begin{array}{l}\because \sin x = \cos\left(\frac{\pi}{2} - x\right) \\ \text{here, } x = \theta - \alpha\end{array}\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \theta + \alpha \quad [\because \cos^{-1}(\cos\theta) = \theta]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \sin^{-1}x + \alpha \quad [\because \theta = \sin^{-1}x]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} + 0$$

$$\left[\because \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$ **(1½)**

**54.** If $y = (\cot^{-1}x)^2$, then show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$

**Delhi 2010C**

To show, $(x^2+1)^2 \frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$

Given, $y = (\cot^{-1} x)^2$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2 \cdot \cot^{-1} x \cdot \frac{d}{dx}(\cot^{-1} x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2 \cot^{-1} x \times \frac{-1}{1+x^2}$$

$$\left[\because \frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}\right]$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = -2 \cot^{-1} x \qquad \textbf{(1½)}$$

Again, differentiating both sides w.r.t. x, we get

$$(1+x^2) \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \times \frac{d}{dx}(1+x^2)$$

$$= \frac{d}{dx}(-2 \cot^{-1} x)$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$$\Rightarrow (1+x^2) \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot 2x = \frac{-2 \times (-1)}{1+x^2} \qquad \textbf{(1½)}$$

$$\left[\because \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}\right]$$

On multiplying both sides by $(1+x^2)$, we get

$$(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2 \qquad \textbf{(1)}$$

**Hence proved.**

**55.** If $y = \text{cosec}^{-1} x, x > 1$, then show that

$$x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0.$$

**All India 2010**

To show, $x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0$

Given, $y = \text{cosec}^{-1} x;\ x > 1$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2 - 1}} \Rightarrow x\sqrt{x^2 - 1} \cdot \frac{dy}{dx} = -1 \quad \textbf{(1)}$$

Again, differentiating both sides w.r.t. x, we get

$$(x\sqrt{x^2 - 1}) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x\sqrt{x^2 - 1}) = \frac{d}{dx}(-1)$$

$$\left[\because \frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}\right] \textbf{(1)}$$

$$\Rightarrow x\sqrt{x^2 - 1} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{x \times \frac{d}{dx}\sqrt{x^2 - 1} + \sqrt{x^2 - 1} \times \frac{d}{dx}(x)\right\} = 0$$

$$\Rightarrow x\sqrt{x^2 - 1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x}{2\sqrt{x^2 - 1}}\frac{d}{dx}(x^2 - 1) + \sqrt{x^2 - 1} \times 1\right\} = 0$$

$$\Rightarrow x\sqrt{x^2 - 1}\frac{d^2y}{dx^2} + \frac{dy}{dx}\left\{\frac{x \cdot 2x}{2\sqrt{x^2 - 1}} + \sqrt{x^2 - 1}\right\} = 0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left\{\frac{x^2}{\sqrt{x^2-1}}+\sqrt{x^2-1}\right\}=0$$

$$\Rightarrow x\sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{x^2}{\sqrt{x^2-1}}\frac{dy}{dx}+\sqrt{x^2-1}\frac{dy}{dx}=0$$

**(1)**

On multiplying both sides by $\sqrt{x^2-1}$, we get

$$x(x^2-1)\frac{d^2y}{dx^2}+x^2\frac{dy}{dx}+(x^2-1)\frac{dy}{dx}=0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2}+(x^2+x^2-1)\frac{dy}{dx}=0$$

$$\Rightarrow \quad x(x^2-1)\frac{d^2y}{dx^2}+(2x^2-1)\frac{dy}{dx}=0$$ **(1)**

**Hence proved.**

**56.** If $y=\cos^{-1}\left(\frac{3x+4\sqrt{1-x^2}}{5}\right)$, then find $\frac{dy}{dx}$.

**All India 2010**

Given, $y = \cos^{-1}\left[\frac{3x + 4\sqrt{1-x^2}}{5}\right]$

Put $x = \sin\theta \Rightarrow \theta = \sin^{-1} x$, we get **(1)**

$$\therefore \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\sqrt{1-\sin^2\theta}}{5}\right]$$

$$\Rightarrow \quad y = \cos^{-1}\left[\frac{3\sin\theta + 4\cos\theta}{5}\right]$$

$$[\because \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta]$$

$$\Rightarrow \quad y = \cos^{-1}\left[\frac{3}{5}\sin\theta + \frac{4}{5}\cos\theta\right]$$ **(1)**

Now, let $\cos\alpha = \frac{4}{5}$ and $\sin\alpha = \frac{3}{5}$

$$\left[\because \sin^2\alpha + \cos^2\alpha = \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1\right]$$

Then, we get

$$y = \cos^{-1}[\sin\theta\sin\alpha + \cos\theta\cos\alpha]$$

$$\Rightarrow \quad y = \cos^{-1}\cos(\theta - \alpha)$$

$$[\because \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta - \alpha)]$$

$$\Rightarrow \quad y = \theta - \alpha$$

$$\Rightarrow \quad y = \sin^{-1} x - \alpha \quad [\because \theta = \sin^{-1} x]$$ **(1)**

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - 0 \quad \left[\because \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}\right]$$

Hence, $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ **(1)**

**57.** Show that the function defined as follows, is continuous at $x=1$, $x=2$ but not differentiable at $x=2$

$$f(x)=\begin{cases}3x-2, & 0<x\le 1\\ 2x^2-x, & 1<x\le 2.\\ 5x-4, & x>2\end{cases}$$

**Delhi 2010**

Firstly, we will check the continuity of the given function at $x=1,2$ and then to check the differentiability of the function $f(x)$ at these points, find LHD and RHD. If LHD $\neq$ RHD, then function is not differentiable.

The given function is $f(x)=\begin{cases}3x-2, & 0<x\le 1\\ 2x^2-x, & 1<x\le 2\\ 5x-4, & x>2\end{cases}$

First, we show the continuity of above function at $x=1$ and at $x=2$.

**Continuity at $x=1$**

$$\text{LHL}=\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(3x-2)$$

[put $x=1-h$, when $x\to 1$, $h\to 0$]

$$\Rightarrow \quad \text{LHL}=\lim_{h\to 0}[3(1-h)-2]$$

$$=\lim_{h\to 0}(3-3h-2)=1$$

$$\text{RHL}=\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2x^2-x)$$

[put $x=1+h$, when $x\to 1$, $h\to 0$]

$$\Rightarrow \text{RHL}=\lim_{h\to 0}[2(1+h)^2-(1+h)]$$

$$=\lim_{h\to 0}[2(1+h^2+2h)-(1+h)]$$

$$=\lim_{h\to 0}[2+2h^2+4h-1-h]$$

$$=\lim_{h\to 0}(2h^2+3h+1) \Rightarrow \text{RHL}=1$$

Also, from the given function, at $x = 1$

$$f(1) = 3(1) - 2 = 3 - 2 = 1$$

Since, $\text{LHL} = \text{RHL} = f(1)$

Hence, $f(x)$ is continuous at $x = 1$. **(1)**

**Continuity at** $x = 2$

$$\text{LHL} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x^2 - x)$$

[put $x = 2 - h$, when $x \to 2, h \to 0$]

$$\Rightarrow \text{LHL} = \lim_{h \to 0} [2(2-h)^2 - (2-h)]$$

$$= \lim_{h \to 0} [2(4 + h^2 - 4h) - (2 - h)]$$

$$= \lim_{h \to 0} (8 + 2h^2 - 8h - 2 + h)$$

$\Rightarrow \text{LHL} = 8 - 2 = 6$ [put $h = 0$]

and $\text{RHL} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (5x - 4)$

[put $x = 2 + h$, when $x \to 2, h \to 0$]

$$\Rightarrow \quad \text{RHL} = \lim_{h \to 0} [5(2 + h) - 4]$$

$$= \lim_{h \to 0} (10 + 5h - 4) = \lim_{h \to 0} (5h + 6)$$

$$\Rightarrow \quad \text{RHL} = 6$$

Also, from the given function, at $x = 2$.

$$f(2) = 2(2)^2 - 2$$

[for $f(2)$, put $x = 2$ in $f(x) = 2x^2 - x$]

$$= 8 - 2 = 6$$

Since, $\text{LHL} = \text{RHL} = f(2)$

So, $f(x)$ is continuous at $x = 2$. **(1)**

Hence, $f(x)$ is continuous at all indicated points.

Now, we verify differentiability of the given function at $x = 1$ and $x = 2$.

**Differentiability at** $x = 1$

$$\text{LHD} = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{[3(1-h) - 2] - [3 - 2]}{-h}$$

$$= \lim_{h \to 0} \frac{(1 - 3h) - (1)}{-h} = \lim_{h \to 0} \frac{-3h}{-h}$$

$$= \lim_{h \to 0} \frac{\ldots}{-h} = \lim_{h \to 0} \frac{\ldots}{-h}$$

$\Rightarrow$ LHD = 3

$$\text{RHD} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{[2(1+h)^2 - (1+h)] - [2-1]}{h}$$

$$= \lim_{h \to 0} \frac{[2 + 2h^2 + 4h - 1 - h] - 1}{h}$$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$$= \lim_{h \to 0} \frac{2h^2 + 3h}{h} = \lim_{h \to 0} (2h + 3)$$

$\Rightarrow$ RHD = 3 [put $h = 0$]

Since, LHD = RHD

So, $f(x)$ is differentiable at $x = 1$. **(1)**

**Differentiability at** $x = 2$

$$\text{LHD} = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$\Rightarrow \text{LHD} = \lim_{h \to 0} \frac{[2(2-h)^2 - (2-h)] - [8-2]}{-h}$$

$$= \lim_{h \to 0} \frac{2(4 + h^2 - 4h) - (2-h) - 6}{-h}$$

$$= \lim_{h \to 0} \frac{h(2h-7)}{-h}$$

$$\Rightarrow \text{LHD} = 7$$

$$\text{RHD} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{[5(2+h) - 4] - [8-2]}{h}$$

$$= \lim_{h \to 0} \frac{(6 + 5h) - (6)}{h}$$

$$= \lim_{h \to 0} \frac{5h}{h}$$

$$\Rightarrow \text{RHD} = 5$$

Since, LHD $\neq$ RHD

So, $f(x)$ is not differentiable at $x = 2$. **(1)**

Hence, $f(x)$ is continuous at $x = 1$ and $x = 2$ but not differentiable at $x = 2$.

**58.** If $y = e^{a \cos^{-1} x}, -1 \leq x \leq 1$, then show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0.$$

**All India 2010**

To show, $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$

Given, $y = e^{a\cos^{-1}x}, -1 \le x \le 1$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = e^{a\cos^{-1}x} \cdot \frac{d}{dx}(a\cos^{-1}x)$$

[by chain rule]

$$\Rightarrow \quad \frac{dy}{dx} = e^{a\cos^{-1}x} \times \frac{-a}{\sqrt{1-x^2}}$$

$$\left[\because \frac{d}{dx}e^x = e^x, \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -ae^{a\cos^{-1}x}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -ay \qquad \ldots(i)$$

[$\because e^{a\cos^{-1}x} = y$, given] **(1)**

Again, differentiating both sides of Eq. (i) w.r.t. x, we get

$$\sqrt{1-x^2} \times \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(\sqrt{1-x^2})$$
$$= \frac{d}{dx}(-ay) \quad [\because \text{ by product rule}]$$

$$\Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2)$$
$$= -a \cdot \frac{dy}{dx}$$

$$\Rightarrow \sqrt{1-x^2} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{-2x}{2\sqrt{1-x^2}} = -a \cdot \frac{dy}{dx}$$

$$\Rightarrow \sqrt{1-x^2}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \cdot \frac{dy}{dx} = -a \cdot \frac{dy}{dx}$$ **(1½)**

On multiplying both sides by $\sqrt{1-x^2}$, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2} \cdot \frac{dy}{dx} \quad \text{...(ii)}$$

But from Eq. (i), we have

$$\sqrt{1-x^2}\frac{dy}{dx} = -ay \Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}} \quad \text{...(iii)}$$ **(1/2)**

On putting the value of $\frac{dy}{dx}$ from Eq. (iii) in Eq. (ii), we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a\sqrt{1-x^2} \times \frac{-ay}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = a^2y$$

$$\therefore (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$ **(1)**

**Hence proved.**

**59.** Find $\frac{dy}{dx}$, if $y = (\cos x)^x + (\sin x)^{1/x}$. **Delhi 2010**

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$. Now, take log on both sides of $u$ and $v$ and then differentiate with respect to $x$ to get $\frac{du}{dx}$ and $\frac{dv}{dx}$. Further, put these values in equation $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$.

Given, $y = (\cos x)^x + (\sin x)^{1/x}$

Let $u = (\cos x)^x$ and $v = (\sin x)^{1/x}$

Then, given equation becomes,

$$y = u + v$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots(i)$$

Now, $u = (\cos x)^x$

On taking log both sides, we get

$$\log u = \log (\cos x)^x$$

$$\Rightarrow \quad \log u = x \log (\cos x)$$

$$[\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}\log (\cos x) + \log (\cos x) \cdot \frac{d}{dx}(x)$$

$$\left[\because \frac{d}{dx}(\log u) = \frac{1}{u}\frac{du}{dx}\right]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{\cos x}(-\sin x) + \log \cos x \cdot 1$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = -x \tan x + \log (\cos x)$$

$$\Rightarrow \quad \frac{du}{dx} = u[-x \tan x + \log \cos x]$$

$$\Rightarrow \quad \frac{du}{dx} = (\cos x)^x$$

$$[-x \tan x + \log \cos x] \quad \ldots(ii) \ \textbf{(1)}$$

Also $v = (\sin x)^{1/x}$

Also, $v = (\sin x)^{1/x}$

On taking log both sides, we get

$$\log v = \log(\sin x)^{1/x}$$

$$\Rightarrow \quad \log v = \frac{1}{x}\log \sin x$$

$[\because \log m^n = n\log m]$

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\frac{d}{dx}(\log \sin x) + \log \sin x\cdot\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{1}{x}\cdot\frac{1}{\sin x}\cdot\cos x + \log \sin x\left(-\frac{1}{x^2}\right)$$

$$\left[\because \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2} \text{ and } \frac{d}{dx}(\log v) = \frac{1}{v}\frac{dv}{dx}\right]$$

$$\Rightarrow \frac{1}{v}\cdot\frac{dv}{dx} = \frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}$$

$$\Rightarrow \quad \frac{dv}{dx} = v\left(\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} = (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right] \quad \text{...(iii)}$$

**(1½)**

On putting the value of $\frac{du}{dx}$ from Eq. (ii) and that of $\frac{dv}{dx}$ from Eq. (iii) in Eq. (i), we get

$$\frac{dy}{dx} = (\cos x)^x[-x\tan x + \log \cos x] + (\sin x)^{1/x}\left[\frac{\cot x}{x} - \frac{\log(\sin x)}{x^2}\right]$$ **(1½)**

**60.** If $y = e^x \sin x$, then prove that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

**All India 2010C**

Firstly, we find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and then put their values along with value of $y$ in LHS of proven expression.

Given, $y = e^x \sin x$ ...(i)

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(e^x)$$

[ by using product rule]

$$\Rightarrow \quad \frac{dy}{dx} = e^x \cdot \cos x + \sin x \cdot e^x \quad \textbf{(1)}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x(\cos x + \sin x) \quad \text{...(ii)}$$

Again, differentiating both sides w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = e^x \cdot \frac{d}{dx}(\cos x + \sin x) + (\cos x + \sin x) \cdot \frac{d}{dx}(e^x)$$

[by using product rule]

$$\Rightarrow \quad \frac{d^2y}{dx^2} = e^x(-\sin x + \cos x) + (\cos x + \sin x) \cdot e^x$$

$$= e^x[-\sin x + \cos x + \cos x + \sin x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2\cos x\, e^x \quad \text{...(iii)} \ \textbf{(1)}$$

Now, we have to show that

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

On putting the values of $\frac{d^2y}{dx^2}$ from Eq. (iii), $\frac{dy}{dx}$ from Eq. (ii) and that of y from Eq. (i) on LHS, we get

$$\text{LHS} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$ **(1)**

$$= 2e^x \cos x - 2e^x (\cos x + \sin x) + 2e^x \sin x$$

$$= 2e^x \cos x - 2e^x \cos x - 2e^x \sin x + 2e^x \sin x$$

$$= 0 = \text{RHS}$$ **(1)**

**Hence proved.**

**61.** If $y = (x)^x + (\sin x)^x$, then find $\frac{dy}{dx}$.

All India 2010C

Given, $y = (x)^x + (\sin x)^x$

Let $\quad u = (x)^x$ and $v = (\sin x)^x$

$\therefore$ Given equation becomes, $y = u + v$

On differentiating w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $\quad u = x^x$ **(1)**

On taking log both sides, we get

$$\log u = \log x^x \Rightarrow \log u = x \log x$$

$$[\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx} = 1 + \log x$$

$$\Rightarrow \quad \frac{du}{dx} = u\,(1 + \log x)$$

$$\Rightarrow \quad \frac{du}{dx} = x^x\,(1 + \log x) \quad [\because u = x^x] \text{ ...(ii)}$$ **(1)**

Also, $\quad v = (\sin x)^x$

On taking log both sides, we get

$$\log v = \log (\sin x)^x$$

$$\Rightarrow \quad \log v = x \log (\sin x) \quad [\because \log m^n = n \log m]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{1}{v}\frac{dv}{dx} = x \cdot \frac{d}{dx}\log(\sin x) + \log(\sin x) \cdot \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x) + \log \sin x$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \log \sin x$$

$$\left[\because \frac{d}{dx}(\log v) = \frac{1}{v}\frac{dv}{dx}\right]$$

$$\Rightarrow \quad \frac{1}{v}\frac{dv}{dx} = x \cot x + \log \sin x$$

$$\Rightarrow \quad \frac{dv}{dx} = v\,(x \cot x + \log \sin x)$$

$$\Rightarrow \quad \frac{dv}{dx} = (\sin x)^x\,(x \cot x + \log \sin x) \quad \text{...(iii)} \quad \mathbf{(1)}$$

On putting the values of $\frac{du}{dx}$ from Eq. (ii) and $\frac{dv}{dx}$ from Eq. (iii) in Eq. (i), we get

$$\frac{dy}{dx} = x^x(1 + \log x)$$
$$+ (\sin x)^x\,(x \cot x + \log \sin x) \quad \mathbf{(1)}$$

**62.** If $y = 3\cos(\log x) + 4\sin(\log x)$, then show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$ **Delhi 2009, 2009C**

To show, $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Given, $y = 3\cos(\log x) + 4\sin(\log x)$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = -3\sin(\log x)\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$$\Rightarrow \quad x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Again, differentiating both sides w.r.t. $x$, we get

$$x \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(x)$$

$$= \frac{d}{dx}[-3\sin(\log x) + 4\cos(\log x)] \quad \textbf{(1½)}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}(\log x)$$

$$\Rightarrow x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\cdot 1 = \frac{-3\cos(\log x)}{x} - \frac{4\sin(\log x)}{x} \quad \textbf{(1½)}$$

$$\Rightarrow x\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-[3\cos(\log x) + 4\sin(\log x)]}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -[3\cos(\log x) + 4\sin(\log x)]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

[$\because$ $3\cos(\log x) + 4\sin(\log x) = y$, given]

Hence, $\quad x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$ **(1)**

**Hence proved.**

**63.** If $y = (x)^{\sin x} + (\log x)^x$, then find $\frac{dy}{dx}$. **Delhi 2009**

Do same as Que 61.

$$\left[\textbf{Ans. } \frac{dy}{dx} = x^{\sin x - 1}[\sin x + x\log x\cos x] + (\log x)^{x-1}[1 + \log x - \log(\log x)]\right]$$

**64.** If $x = a\left(\cos\theta + \log\tan\frac{\theta}{2}\right)$ and $y = a\sin\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$. **Delhi 2009C**

Do same as Que 6.

$$\left[\textbf{Ans. } \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{2\sqrt{2}}{a}\right]$$

**65.** If $y = (\log x)^x + (x)^{\cos x}$, then find $\frac{dy}{dx}$.

**Delhi 2009C**

Do same as Que 61.

$$\left[\textbf{Ans. } (\log x)^{x-1}[1+\log x \cdot \log(\log x)] + x\cos^{x-1}[\cos x - x\log x \sin x]\right]$$

**66.** If $y = e^x (\sin x + \cos x)$, then show that

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$

**All India 2009**

Given, $y = e^x (\sin x + \cos x)$ ...(i)

To show, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ ...(ii)

On differentiating both sides of Eq. (i) w.r.t. $x$, we get

$$\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(\sin x + \cos x)$$

$$+ (\sin x + \cos x) \cdot \frac{d}{dx}(e^x)$$

$$= e^x (\cos x - \sin x) + (\sin x + \cos x) \cdot e^x$$

$= e^x [\cos x - \sin x + \sin x + \cos x]$

$= e^x (2 \cos x)$

$\Rightarrow \quad \frac{dy}{dx} = 2e^x \cos x$ ...(iii) **(1½)**

Again differentiating both sides of Eq. (iii) w.r.t. $x$, we get

$$\frac{d^2y}{dx^2} = (2e^x) \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(2e^x)$$

$$= 2e^x (- \sin x) + \cos x \cdot 2e^x$$

$$= 2e^x \cos x - 2e^x \sin x \quad \text{...(iv)}$$ **(1½)**

Now, we put the values of $\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ from Eqs. (iv) and (iii) along with value of $y$ in LHS of Eq. (ii), we get

$$\text{LHS} = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y$$

$$= (2e^x \cos x - 2e^x \sin x) - 2(2e^x \cos x) + 2e^x (\sin x + \cos x)$$

$$= 2e^x \cos x - 2e^x \sin x - 4e^x \cos x + 2e^x \sin x + 2e^x \cos x$$

$$= 4e^x \cos x - 4e^x \cos x = 0 = \text{RHS}$$ **(1)**

**Hence proved.**

**67.** If $y = \frac{\sin^{-1} x}{\sqrt{1 - x^2}}$, then show that

$$(1 - x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0.$$

**All India 2009**

Given, $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2} \times \frac{d}{dx}(\sin^{-1} x) - (\sin^{-1} x) \times \frac{d}{dx}\sqrt{1-x^2}}{(\sqrt{1-x^2})^2}$$

$$= \frac{\left[\sqrt{1-x^2} \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot \frac{d}{dx}(1-x^2)\right]}{(\sqrt{1-x^2})^2} \quad \textbf{(1)}$$

$$= \frac{\left[\sqrt{1-x^2} \cdot \frac{1}{\sqrt{1-x^2}} - (\sin^{-1} x) \cdot \frac{-2x}{2\sqrt{1-x^2}}\right]}{1-x^2}$$

$$= \frac{1 + \frac{x \sin^{-1} x}{\sqrt{1-x^2}}}{(1-x^2)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+xy}{1-x^2} \qquad \left[\because \frac{\sin^{-1} x}{\sqrt{1-x^2}} = y, \text{ given}\right]$$

$$\Rightarrow \quad (1-x^2)\frac{dy}{dx} = 1 + xy \qquad \textbf{(1)}$$

Again differentiating above equation both sides w.r.t. $x$, we get

$$(1-x^2) \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}(1-x^2) = \frac{d}{dx}(1+xy)$$

**(1)**

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) = x\frac{dy}{dx} + y \cdot 1$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0 \qquad \textbf{(1)}$$

**Hence proved.**

**68.** Differentiate the following function w.r.t. $x$.

$(x)^{\cos x} + (\sin x)^{\tan x}$ **Delhi 2009**

Do same as Que 61.

$$\left[\textbf{Ans. } x^{\cos x - 1}[\cos x - x \log \sin x] + (\sin x)^{\tan x}[1 + \sec^2 x \log \sin x]\right]$$

**69.** Differentiate the following function w.r.t. $x$.

$x^{\sin x} + (\sin x)^{\cos x}$ **Delhi 2009**

Do same as Que 61.

$$\left[\textbf{Ans. } x^{\sin x - 1}[\sin x + x \log x \cos x] + \sin x^{\cos x}[\cos x \cot x - \sin x \log(\sin x)]\right]$$

**70.** If $y = x^{\cot x} + (\sin x)^x$, then find $\frac{dy}{dx}$.

**All India 2008C**

Do same as Que 61.

$$\left[\textbf{Ans. } x^{\cot x-1}[\cot x - x\log x \operatorname{cosec}^2 x] + (\sin x)^x[x\cot x + \log x \sin x]\right]$$

**71.** If $xy + y^2 = \tan x + y$, then find $\frac{dy}{dx}$.

**All India 2008C**

Firstly, differentiate the given expression with respect to $x$ and then collect all the first derivative of $y$ on one side to get the required result.

Given equation is $xy + y^2 = \tan x + y$

On differentiating both sides w.r.t. $x$, we get

$$\frac{d}{dx}(xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x + y) \quad \textbf{(1/2)}$$

$$\Rightarrow \left[x \cdot \frac{d}{dx}(y) + y \cdot \frac{d}{dx}(x)\right] + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx}$$

**(1½)**

$$\Rightarrow x \cdot \frac{dy}{dx} + y \cdot 1 + 2y\frac{dy}{dx} = \sec^2 x + \frac{dy}{dx} \quad \textbf{(1)}$$

$$\Rightarrow (x + 2y - 1)\frac{dy}{dx} = \sec^2 x - y$$

$$\therefore \frac{dy}{dx} = \frac{\sec^2 x - y}{x + 2y - 1} \quad \textbf{(1)}$$

**72.** If $y = (\log x)^{\cos x} + \frac{x^2+1}{x^2-1}$, then find $\frac{dy}{dx}$.

**Delhi 2008C**

Do same as Que 37.

$$\left[\textbf{Ans. } (\log x)^{\cos x}\left[\frac{\cos x}{x\log x} - \sin x \log(\log x)\right] - \frac{4x}{(x^2-1)^2}\right]$$

**73.** If $y = \sin^{-1}\left[\frac{5x + 12\sqrt{1 - x^2}}{13}\right]$, then find $\frac{dy}{dx}$.

**HOTS; All India 2008**

Do same as Que 56. $\left[\textbf{Ans. } \frac{1}{\sqrt{1 - x^2}}\right]$

**74.** If $y = \cot^{-1}\left[\frac{\sqrt{1 + \sin x} + \sqrt{1 - \sin x}}{\sqrt{1 + \sin x} - \sqrt{1 - \sin x}}\right]$, $0 < x < \frac{\pi}{2}$, then find $\frac{dy}{dx}$.

**HOTS; Delhi 2008**

Firstly, convert the given function into simplest form by using

$$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2$$

and $1-\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}$

and then differentiate.

Given, $y=\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(i)

Now, on putting

$$1+\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}$$

and $1-\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}$

$-2\sin\frac{x}{2}\cos\frac{x}{2}$ in Eq. (i), we get **(1)**

$$y=\cot^{-1}\left[\frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}+\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}-\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}\right]$$

$$\Rightarrow y=\cot^{-1}\left[\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right]$$

**(1)**

$$\left[\because a^2+b^2+2ab=(a+b)^2\right]$$

$$\left[\begin{array}{l}\text{and } a^2 + b^2 - 2ab = (a-b)^2 \\ \text{and here, } a = \cos\frac{x}{2},\ b = \sin\frac{x}{2}\end{array}\right]$$

$$\Rightarrow y = \cot^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]$$

$$\Rightarrow \quad y = \cot^{-1}\left[\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right] \qquad \textbf{(1)}$$

$$\Rightarrow \quad y = \cot^{-1}\cot\frac{x}{2} \qquad \left[\because \cot\theta = \frac{\cos\theta}{\sin\theta}\right]$$

$$\Rightarrow \quad y = \frac{x}{2} \qquad [\because \cot^{-1}\cot\theta = \theta]$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{1}{2} \qquad \textbf{(1)}$$

**NOTE** (i) When $0 < x < \frac{\pi}{2}$, then we consider

$$\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}$$

(ii) When $\frac{\pi}{2} < x < \pi$, then we consider

$$\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \sqrt{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)^2}$$

**75.** If $y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$, then find $\frac{dy}{dx}$.

**Delhi 2008**

Given, $y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right)$

It can be written as

$$y = \sqrt{x^2+1} - \log\left(\frac{1}{x} + \sqrt{\frac{x^2+1}{x^2}}\right)$$

$$\Rightarrow \quad y = \sqrt{x^2+1} - \log\left(\frac{\sqrt{x^2+1}+1}{x}\right) \quad \textbf{(1/2)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2+1}\right) - \frac{d}{dx}\log\left(\frac{\sqrt{x^2+1}+1}{x}\right)$$

$$= \frac{1}{2\sqrt{x^2+1}}\frac{d}{dx}(x^2+1) - \frac{d}{dx}\log\left(\frac{\sqrt{x^2+1}+1}{x}\right)$$

$$= \frac{2x}{2\sqrt{x^2+1}} - \frac{1}{\left(\frac{\sqrt{x^2+1}+1}{x}\right)} \times \frac{d}{dx}\left(\frac{\sqrt{x^2+1}+1}{x}\right) \quad \textbf{(1)}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1} \times \frac{x \cdot \frac{d}{dx}(\sqrt{x^2+1}+1) - (\sqrt{x^2+1}+1)\frac{d}{dx}(x)}{x^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{x\left(\frac{2x}{2\sqrt{x^2+1}}\right) - (\sqrt{x^2+1}+1)\cdot 1}{x^2}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{\frac{x^2}{\sqrt{x^2+1}} - (\sqrt{x^2+1}+1)}{x^2}$$

$$= \frac{x}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}+1}$$

$$\times \frac{x^2 - (\sqrt{x^2+1})(\sqrt{x^2+1}+1)}{x^2(\sqrt{x^2+1})}$$

$$= \frac{x}{\sqrt{x^2+1}} - \left[\frac{x^2 - x^2 - 1 - \sqrt{x^2+1}}{x(\sqrt{x^2+1}+1)(\sqrt{x^2+1})}\right]$$

**(1½)**

$$= \frac{x}{\sqrt{x^2+1}} - \frac{(-\sqrt{x^2+1}-1)}{x(\sqrt{x^2+1}+1)(\sqrt{x^2+1})}$$

$$= \frac{1}{\sqrt{x^2+1}}\left[x + \frac{(\sqrt{x^2+1}+1)}{x(\sqrt{x^2+1}+1)}\right]$$

$$= \frac{1}{\sqrt{x^2+1}}\left(x + \frac{1}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} \cdot \frac{(x^2+1)}{x}$$

Hence, $$\frac{dy}{dx} = \frac{\sqrt{x^2+1}}{x}$$ **(1)**

**76.** Differentiate the following function w.r.t. $x$.

$$\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$ **Delhi 2008**

Reduce the given function into simplest form by putting $x = \cos\theta$ and by using the half angle formulae

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2} \text{ and } 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

Then, find its derivative with respect to x.

Given function is

$$y = \tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right]$$

On putting $x = \cos\theta$

$\Rightarrow \quad \theta = \cos^{-1} x$, we get

$$\therefore \quad y = \tan^{-1}\left[\frac{\sqrt{1+\cos\theta} - \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}\right] \quad \text{...(i)} \; \textbf{(1)}$$

We know that,

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

and

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

On putting the above values in Eq. (i), we get

$$y = \tan^{-1}\left[\frac{\sqrt{2\cos^2\frac{\theta}{2}} - \sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}} + \sqrt{2\sin^2\frac{\theta}{2}}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right) \quad \textbf{(1)}$$

On dividing numerator and denominator by $\cos\frac{\theta}{2}$, we get

$$y = \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right)$$

$$\left[\because \quad 1 = \tan\frac{\pi}{4} \quad \text{and } \tan\frac{\theta}{2} = 1\times\tan\frac{\theta}{2} = \tan\frac{\pi}{4}\tan\frac{\theta}{2}\right]$$

$$\Rightarrow \quad y = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$\left[\because \frac{\tan A - \tan B}{1+\tan A \tan B} = \tan(A-B)\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4}-\frac{\theta}{2} \qquad [\because \tan^{-1}\tan x = x]$$

$$\Rightarrow \quad y = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \qquad [\because \theta = \cos^{-1}x] \quad \textbf{(1)}$$

On differentiating both sides w.r.t. $x$, we get

$$\frac{dy}{dx} = 0 - \frac{1}{2}\times\frac{-1}{\sqrt{1-x^2}} \left[\because \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}\right]$$

Hence, $\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$ **(1)**